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Nonsmooth Impact Mechanics
Models, Dynamics and Control

Springer
Preface

This monograph is devoted to the study of a class of dynamical systems of the general form

\[ \dot{x} = g(x, u) \]
\[ f(x, t) \geq 0 \]  

(0.1)

where \( x \in \mathbb{R}^n \) is the system's state vector, \( u \in \mathbb{R}^m \) is the vector of inputs, and the function \( f(\cdot, \cdot) \) represents a unilateral constraint that is imposed on the state. More precisely, we shall restrict ourselves to a subclass of such systems, namely mechanical systems subject to unilateral constraints on the position, whose dynamical equations may be written as

\[ \ddot{q} = g(q, \dot{q}, u) \]
\[ f(q, t) \geq 0 \]  

(0.2)

where \( q \in \mathbb{R}^n \) is the vector of generalized coordinates of the system and \( u \) is an input (or controller) that generally involves a state feedback loop, i.e. \( u = u(q, \dot{q}, t, z) \), with \( \dot{z} = Z(z, q, \dot{q}, t) \) when the controller is a dynamic state feedback. Mechanical systems composed of rigid bodies interacting fall into this subclass. A general property of systems as in (0.1) and (0.2) is that their solutions are nonsmooth (with respect to time): Nonsmoothness arises from the occurrence of impacts (or collisions, or percussions) in the dynamical behaviour, when the trajectories attain the surface \( f(x, t) = 0 \). They are necessary to keep the trajectories within the subspace \( \Phi = \{ x : f(x, t) \geq 0 \} \) of the system's statespace. It is therefore necessary, when dealing with such classes of dynamical systems, to focus on collisions dynamics.

Impact phenomenons between perfectly rigid bodies have been studied by scientists since a long time. It is worth noting that the problems related to impact dynamics have attracted the interest of physicists for at least 3 centuries. The impact physical laws were in particular studied and used initially by scientists like Newton [397] [88], Poisson [440], Huygens [225], Coriolis [113], the well-known Newton's and Poisson's restitution coefficients being still well alive as basic models for rigid bodies collisions, and later by Darboux [120], Routh [461], Appell [14] and others [583] [388] [321]. Although this fact has been a little forgotten now, rigid body (or more exactly particles) shock dynamics were extensively used in the seventeenth century to study light models [225] and also by artillerists [333] to predict the flight of cannon balls and their impacts. They are currently still used to describe motion of molecules in ideal gas [514] [105], in relationship with complex dynamical behaviour.
of so-called billiards\(^1\). Mathematicians (for problems related to existence of solutions to specific dynamical problems, analysis of complex dynamics of certain impacting systems like billiards\(^2\)), researchers from mechanical engineering, robotics (to study the effect of impacts in the joints or the motion of the system after the impact, like in robot manipulators, bipede, juggling or hopping robots, multifingered hands, \ldots), computer sciences (graphics, virtual reality) have also been interested (and are still) by rigid body dynamics with shocks (or more exactly, as we shall see throughout this book, with unilateral constraints).

The interest of scientists from so different horizons for rigid bodies dynamics and unilateral constraints (which include impact dynamics) can be explained from the fact that, as we shall see, rigidity allows simplification of the dynamical contact-impact problem (an important point for engineers who have to design systems subject to percussions), but at the same time involves some deep mathematical problems (related to existence, uniqueness and approximating procedures of solutions of problems with unilateral constraints, or nonsmooth dynamics). Rigid body dynamics also yield strong mechanical problems, as it is well-known in mechanisms theory that rigidity may sometime create some sorts of undetermined problems (with no or several solutions). It can then become important (think of graphics and virtual reality) to possess fast enough numerical algorithms capable of providing "believable" motions (i.e. solutions not too far from what most of the experiments show).

It seems that dynamics of mechanical systems with unilateral constraints have been a little neglected in the general mechanical literature (in general, the textbooks contain -when they do- a single chapter or paragraph where impacts and/or friction dynamics are treated in a rather rapid manner. In a survey, Barmes [34] reviewed 69 textbooks containing a section on collisions. According to him, only 6 of them were theoretically sound, and only one was acceptable [442]. Certainly he was not aware of Pérès' book [429], although published before the survey was written, and of other books not written in English). One of the main reasons for this situation must be the algebraic form of shock dynamics between rigid bodies. These algebraic equations have led many people in the mechanical engineering field to consider impact dynamics as a very simple, hence not very interesting topic. One of the goals of this monograph is to prove that it is rather the contrary which is true: Dynamics of systems with unilateral constraints are far from yielding simple analysis and require sophisticated mathematical tools to be well understood. On the other hand, shocks and friction macroscopic models are still an active research area.

\(^1\)In the literature, it seems that the word vibro-impact systems is used in the mechanical engineering field to name various types of systems that involve percussions. The word billiards refers to theoretical models of particles colliding in a closed domain, and is used mainly in mathematical physics.

\(^2\)There has been in the past decades a great mathematical effort devoted to the study of billiards in relationship with Boltzmann's conjecture on the stochastic properties of motion of elastic balls in a closed domain, with elastic collisions, see e.g. the celebrated works of the Russian mathematician Sinai [493] [494], see also the book [292], and references therein. The study of billiards started with Coriolis [113].
The study of dynamical systems subject to impacts can be split into different classes, that more or less reflect the interest of the different groups of researchers listed above (Historically, contact-impact problems have always been studied by analytical, numerical and experimental methods [404]). Among them one may find the following ones:

i) Study the well-posedness of impact dynamical equations, i.e. properties of solutions (existence, uniqueness, continuous dependence on initial conditions and parameters, stability ...) of differential equations (or inclusions) containing measures, that model contact percussions. Distributions and measures theory is at the core of this studies. This may also include the study of convergence of smooth problems towards nonsmooth ones, as well as the variational formulation of percussion dynamics.

ii) Study the global motion of systems subject to unilateral constraints (The preceding step being necessary but far from sufficient to solve this one). This includes in particular investigation of complex nonsmooth dynamics of multi-body systems with unilateral constraints, with particular emphasis on uniqueness of the solution, problems of simultaneity and possible break of contacts.

iii) Study the impact between two rigid bodies via macroscopic laws that relate the motion after and before the shocks, without explicit model of contact in the compliant bodies case, in particular extend basic laws such as Newton's or Poisson's restitution laws when there is friction and slip during the impact. Study the extension of restitution rules for complex mechanical systems submitted to several unilateral constraints.

iv) Develop new sophisticated models of contact-impact laws based on Hertz, Cattaneo-Mindlin theories, finite element methods...

v) Study dynamics of systems subject to impacts (like the bouncing-ball as a benchmark example, or the linear oscillator) as discrete-time systems (Poincaré maps), the instants of discretization being the impact times. The core of these works is the generally complex behaviour that result from combination of very simple continuous and discrete dynamics.

vi) Investigate the complex dynamics (periodic trajectories, bifurcations, chaotic behaviour) of vibro-impact systems like the impact oscillator, that may model systems like mechanisms with clearance (In machine-tools, space structures...), or of billiards.

vii) Study numerical algorithms to integrate systems subject to unilateral constraints.

viii) Develop experimental devices and systems to substantiate the analytical models and predictions.

3This list may not be exhaustive.
ix) Design control strategies to improve the behaviour of mechanical systems subject to repeated impacts with an environment (impacts are here considered as disturbances), or on the contrary use "percussive controllers" to stabilize a system.

The primary objective in doing this work was about control of mechanical systems with unilateral constraints, and more expressly extension of classical schemes on hybrid force/position control of manipulators submitted to holonomic frictionless constraints [324] [606]. As noted in [72], "...it is an oversimplification to imagine that the goal of a robot is simply to traverse a prescribed path in position and orientation space., and impacts play a major role in many robotic manipulation tasks. Also, the contact problem is unsolved for rigid manipulator, rigid sensor, rigid environment problems. [422], and a necessary step towards a solution is a clear understanding of percussion dynamics. The subject may well possess other important applications as we noted above, like bipedal locomotion, which has become an important topic of investigations within the mechanical engineering and robots control communities, and whose dynamics are generally based on rigid bodies impacts assumptions. Also the control of all mechanisms with clearances taking into account impact dynamics is a topic deserving future works. At some places of the book applications of impact dynamics are mentioned. Consequently, this work aims at gathering and if possible comparing different works that concern rigid percussion dynamics, and also at giving an overview as clear and as complete as possible on the very nature of rigid body impact dynamics. It is clear that mathematicians will not learn much mathematics here, and that mechanicians will not learn much mechanics! But I hope that those who desire to get acquainted with this topic and get a general point of view on it, will find what they are looking for, whatever their background may be.

Some topics (points iv, vi, viii) are not or only partially covered in this monograph. Point vii is rapidly commented. The interested reader can consult concerning iv the papers [243] [244] [614] where more than 500 references can be found on the subject. A survey concerning point vi on systems with clearances can be found in [184], although many studies on dynamics of such systems (points v, vi, vii) have been done since that time. Here only a small part of point vi will be treated, concerning existence and stability of periodic trajectories. I admit that some place could have been devoted these problems, and that they certainly are of interest to the control of mechanical systems. One therefore realizes how numerous and various the problems involving impacts are, and that they can hardly be treated all at the same time!

The work rather focuses on points i, ii, iii, v and ix, and I try to introduce the different works that have been devoted to this field in a logical way. Also since the goal is to provide solid theoretical bases for control theory of impacting mechanical systems, I insist on the mathematical sides of the subject (which is a necessary step to study stability of such systems and extend previous works on control of constrained manipulators [324] [606]). A state-of-the-art of percussion models that may be useful in this context is also presented.
To the best of my knowledge it is the first time that all the presented works are gathered and discussed. This monograph is therefore to be considered as what should be the first version of a more detailed and complete work, making all the various sides of nonsmooth dynamics accessible in one volume. I am therefore open to all the remarks and comments one might have on my work. Although this might not be too important, let us mention that around 90 references have been found in the journals ASME J. of Applied Mechanics (34 papers), PMM USSR translated in J. of Applied Mathematics and Mechanics (30 papers) and the journal of Sound and Vibrations (24 papers), which seem to provide the most important source of references on rigid body impact dynamics (around 15 percent of the total number of references, but a much higher percentage of journal papers). Also 22 papers from the American Journal of Physics (dedicated to teachers) prove that impact dynamics are present in teaching. From my own recent experience I feel that there are some "gaps to fill" between researchers from very different horizons that work on the same topic. Control theory and practice of mechanical systems with unilateral constraints may be a domain requiring knowledge from all the other parts, hence the necessity to have a general point of view on this topic. Finally the interested reader is invited to have a look at some other recently published monographs with related contents: The book of R.M. Brach [54] is oriented towards rigid body impacts and Newton like restitution rules, sustained by many experimental results, and belongs to the Mechanical Engineering field. In [292], V.V. Kozlov and D.V. Treshchev provide a detailed analysis of billiards dynamics. In [366], M. Monteiro-Marques deals with existential results for the sweeping process evolution problem. Both monographs [292] and [366] belong to the Applied Mathematics field, although the first one is oriented towards dynamical systems analysis (see also parts of [262] and [181] dedicated to that field), whereas the second one belongs to the functional analysis field. Let us also mention the book by D.D. Bainov and P.S. Simeonov [27] which deals with stability of dynamical systems submitted to impulses, although the systems analyzed in [27] do not fit with systems subject to unilateral constraints (mainly the authors analyze systems subject to impulses at fixed exogeneous instants). Among older books on mechanics, the ones by Pérès [429] and by Moreau [376] contain an important chapter dedicated to rigid body impacts. Many other books contain a part devoted to this topic, and it is neither possible nor worth citing them here. There are also several monographs on impact dynamics and vibro-impact systems in the Russian literature, whose references can be found in most of the bibliographical notes of the cited translated Russian papers.

Acknowledgements

This work would not have been possible without the help and the numerous precious discussions I have had during the past two years on nonsmooth and impact...
dynamics with the following persons:

- Manuel Monteiro-Marques, Centro de Matematica e Aplicações Fundamentais, Universidade de Lisboa, Portugal.
- Ray Brach, University of Notre-Dame, Aerospace and Mechanical Engineering Dept., USA.
- Jean-Jacques Moreau, Laboratoire de Mécanique et Génie Civil, Université de Montpellier, France.
- Alexander Ivanov, Moscow State Academy of Instrument Making, Russia.

I would like also to thank very much Professors T. Küpper (Mathematisches Institut, Köln) and K. Popp (Institut für Mechanics, Hannover), who invited me to the 141-WE-Heraeus-Seminar, "Analysis of non-smooth dynamical systems", Physikzentrum Bad Honnef, march 1995, Germany. May I also thank Prof. K. Deimling (University of Paderborn, Germany), for the discussions we had there, and the encouragements he gave me to write this monograph. I would like also to thank L. Paoli and M. Schatzman (Université C. Bernard, Lyon, France), Y. Hurmuzlu (Southern Methodist Univ., USA), M. Jean (LMGC, Montpellier, France), F. Pierrot (LIRMM, Montpellier), A. Goswamy (INRIA, Grenoble), Rogelio Lozano (UT Compiègne), S. Niculescu, P. Orhant, A. Zavala, M. Mata for the discussions we have had on impact dynamics and control, and my colleagues of the Robotic group of the LAG for help concerning MacDrawPro and LateX.

Saint-Martin d’Hères, April 1996
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Chapter 1

Distributional model of impacts

This chapter is devoted to introduce the mathematical basis on which nonsmooth impact dynamics rely. Emphasis is put on measure differential equations (MDE), and on the difference between several types of MDE's and those which represent the dynamics of mechanical systems with unilateral constraints. Variable changes that allow to transform such MDE's into Carathéodory ordinary differential equations are presented.

1.1 External percussions

In this section, we introduce the impacts as purely exogeneous actions on a mechanical system, without considering the way by which they may be produced. Simply speaking, an impact between two bodies (not necessarily rigid) is a phenomenon of very short duration, that implies a sudden change in the bodies dynamics (velocity jump) and generally (but not necessarily) involves a large interaction wrench at the instant of the collision. In most of the robotics and mechanical engineering literature on the subject impacts are treated as very large forces acting during an infinitely short time, i.e. if $\Delta t$ represents the collision duration and $p(\tau)$ represents the force during the collision ($p(\cdot)$ may be viewed as a time function whose support is $K = [t_k, t_k + \Delta t]$, i.e. $p(\cdot)$ is zero outside $K$), then the force impulse $p_i$ due to the impact at time $t_k$ is:

$$p_i = \lim_{\Delta t \to 0} \int_{t_k}^{t_k + \Delta t} p(\tau) d\tau$$

In order for the right-hand-side of (1.1) to be a nonzero quantity, the integrand $p(\tau)$ must take infinite values, as the integration interval is of zero measure. It therefore follows that $p(\cdot)$ cannot be a function of time (it is almost everywhere zero and its Lebesgue integral is not zero [478]), and must be considered as a (singular) distribution or Dirac measure at time $t_k$, denoted as $\delta_{t_k}$, with magnitude $p_i$, i.e. $p \equiv p_i \delta_{t_k}$. It is worth noting that this is not just one way to represent the percussion forces according to equation (1.1), but this is the only formulation of such phenomena that is mathematically correct: "Analogy between mathematical and physical distributions.
2 CHAPTER 1. DISTRIBUTIONAL MODEL OF IMPACTS

Figure 1.1: Mass submitted to an external percussion.

has not to be shown: mathematical distributions provide a correct mathematical definition of distributions encountered in physics", [479] (Chapter 1, p.84). Therefore if one chooses to represent impacts following this philosophy (see (1.1)), the dynamical equations of the considered bodies have to be treated in the sense of Schwartz’s distributions. We think it is important to reiterate that it has indeed to be used in order to correctly represent these phenomena.

One of the main consequences of such an approach is that the impacts imply a discontinuity in the velocity. This can be easily understood from simple examples:

Example 1.1 Assume that a mass \( m \) moving on a line, with gravity center coordinate \( x \) (the system is depicted in figure 1.1), is submitted to an impact of magnitude \( p_i \) at the instant \( t_k \). We have:

\[
m\ddot{x} = p_i \delta_{t_k}
\]

(1.2)

Assume now that \( x \) and \( \dot{x} \) possess (possibly zero) respective jumps \( \sigma_x = x(t_k^+) - x(t_k^-) \) and \( \sigma_{\dot{x}} = \dot{x}(t_k^+) - \dot{x}(t_k^-) \) at \( t_k \). In the following we shall prove that if \( p_i \) is not zero, then \( \sigma_{\dot{x}} \) is neither, whereas \( \sigma_x = 0 \). We have:

\[
\ddot{x} = \{\ddot{x}\} + \sigma_{\dot{x}} \delta_{t_k}
\]

(1.3)

\[
\ddot{x} = \{\ddot{x}\} + \sigma_x \delta_{t_k} + \sigma_{\dot{x}} \delta_{t_k}
\]

where \( \{f\} \) represents the derivative of \( f \) calculated ignoring the points of discontinuity, and which is not defined at the points of discontinuity [478] (chapter 2, section 3). For instance the distributional derivative of the Heaviside function \( h(t) \equiv 0 \) for \( t < t_k, h(t) \equiv 1 \) for \( t \geq t_k \) is \( \dot{h} = \{h\} + \delta_{t_k} = 0 + \delta_{t_k} = \delta_{t_k} \). The notation \( Dh \) instead of \( \dot{h} \) is generally used to denote the distributional derivative of a function \( h \) [477]. For instance (1.2) should be written \( DX = AX + p_i DH \), with \( X^T = (x, \dot{x}) \), \( H^T = (0, h) \) and \( A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). Such differential equations are called measure differential equations, since the distributional derivative \( Df \) of any function \( f \) of bounded variation can be identified with a Stieltjes measure, and conversely a Stieltjes measure defined
by \( h \) above is a Dirac distribution \([178]\) p.132, \([478]\) theorem 2, chapter 2. Some basic facts about distributions and measures are recalled in appendix A. We shall come back in detail on such measure differential equations and their relationships with dynamical equations of mechanical systems subject to unilateral constraints in this chapter. Note also that the procedure we employ here makes use of the derivative of the Dirac measure, which is not a measure \([478]\). This justifies again that we choose Schwartz’s distributions as an analytical tool for our study, although as pointed out in \([385]\] the analysis of nonsmooth dynamics for collisions rests only on measures since it involves only signed distributions. We choose here the notation employed in \([478]\). Then introducing (1.3) into (1.2) we get:

\[
m\{\dot{x}\} = p_t \delta_{t_k} - m\sigma_x \dot{\delta}_{t_k} - m\sigma_z \delta_{t_k}
\]

Consider now (1.4). On \([t_0, t_k)\), \( t_0 < t_k \), the system has a smooth solution \( x(t), \dot{x}(t) \). Thus \( m\{\dot{x}\} \) has support \( K_1 \) contained in \([t_0, t_k)\). The right-hand-side of (1.4) has support \( K_2 = t_k \). Then we conclude that the only way to have (1.4) verified (i.e. \( m\{\dot{x}\} - (p_t - m\sigma_x) \delta_{t_k} + m\sigma_z \dot{\delta}_{t_k} = 0 \)) is that \( m\{\dot{x}\} = 0 \) and \((p_t - m\sigma_x) \delta_{t_k} + m\sigma_z \dot{\delta}_{t_k} = 0\), because these two distributions must take the same value on any function \( \varphi \in \mathcal{D} \) whose support does not contain \( t_k \), i.e. zero. Recall that these equalities have to be taken in the sense of distributions, and that the value of the function \( \{\dot{x}\} \) at \( t = t_k \) needs not to be specified, as almost-everywhere equal functions define the same distribution. Now we are left with \((p_t - m\sigma_x) \delta_{t_k} + m\sigma_z \dot{\delta}_{t_k} = 0\). If these two singular distributions were equal, we should get for any function \( \varphi \in \mathcal{D} \) with support \( K_{\varphi} \) containing \( t_k \): \(<(p_t - m\sigma_x) \delta_{t_k} + m\sigma_z \dot{\delta}_{t_k}, \varphi> = (p_t - m\sigma_x) \varphi(t_k) + m\sigma_z \dot{\varphi}(t_k) = 0\) \(^1\). Take \( \varphi_{t_k} \equiv \varphi(t - t_k) \) where \( \varphi \) is defined in (A.1) (see appendix A), and note that \( \frac{d}{dt} \varphi_{t_k}(t_k) = 0 \): we obtain \( p_t - m\sigma_x(t_k) = 0 \). Thus \( m\sigma_z(t_k) = 0 \) as well. Note that we could have also taken two functions \( \varphi_1, \varphi_2 \in \mathcal{D} \) such that the matrix

\[
A \equiv \begin{pmatrix}
\varphi_1(t_k) & \dot{\varphi}_1(t_k) \\
\varphi_2(t_k) & \dot{\varphi}_2(t_k)
\end{pmatrix}
\]

is full-rank. Then one gets \( A \begin{pmatrix} p_t - m\sigma_x(t_k) \\ m\sigma_z(t_k) \end{pmatrix} = 0 \), which implies that both components are zero. Hence we get:

\[
\begin{align*}
m\{\dot{x}\} &= 0 \\
p_t - m\sigma_x(t_k) &= 0 \\
m\sigma_z(t_k) &= 0
\end{align*}
\]

from which we conclude \( \sigma_x(t_k) = 0 \) (\( x \) is continuous at \( t_k \)), \( \sigma_z(t_k) = \frac{p_t}{m} \) (\( \dot{x} \) jumps at \( t_k \)). From (1.5) we deduce that \( \dot{x} = \dot{x}(t_k^-) = \dot{x}_0 \) for \( t < t_k \), \( \dot{x} = \dot{x}(t_k^+) = \dot{x}_0 + \frac{p_t}{m} \) for \( t \geq t_k \), \( \dot{x} \) is zero almost everywhere and is a Dirac measure of magnitude \( \frac{p_t}{m} \) at \( t_k \), \( x(t) = x_0 + x_0 \) for \( t < t_k \), \( x(t) = (x_0 + \frac{p_t}{m}) t - \frac{p_t}{m} t + x_0 \) for \( t \geq t_k \). The equalities in (1.5) are therefore necessary for (1.4) to be true. Sufficiency is straightforward. Thus we have proved the following:

\(^1\)Recall that distributions are indefinitely differentiable, and that the derivatives of the Dirac measure are defined as \(<\delta_{t_k}^{(m)}, \varphi> = (-1)^m \varphi^{(m)}(t_k)\) for \( m \geq 0 \), \([478]\).
Claim 1.1 Assume the mass is submitted to an impulsive percussion \(^2\) at \(t = t_k\). Then there is a discontinuity \(\sigma(x)\) in the velocity at the time \(t_k\) while the position \(x\) remains continuous. Conversely if the velocity is discontinuous and the position continuous, then there is an impulsive percussion.

In the literature, the analysis generally starts by \textit{a priori} assuming that during the impact, the change in the bodies' position can be neglected, or that \textit{rigid body collisions can be considered as a time-dependent process where changes in momenta occur without changes in configuration since the total collision period is very small [519]: the formulation given here provides a rigorous explanation of this assumption. Moreover the use of Dirac measures to model percussions clarifies the notion of infinitely large force acting during a very short time, as it allows one to separate the magnitude of the contact percussion \(p\) and its distribution on the time axis \(\delta(t)\). Let us fix the following definitions \([385]\(^3\):

\textbf{Definition 1.1} A \textit{force} \(F(t)\) acting on a system is the density with respect to the Lebesgue measure \(dt\) of the \textit{contact impulsion measure} \(dP\), i.e. \(P(t) = \int_{t_0}^{t} F(x)dx\).

\textbf{Definition 1.2} A \textit{contact percussion} is an atom at the impact time \(t_k\) of the \textit{contact impulsion measure} \(dP\). The \textit{percussion vector} \(p\) is the density of the atom with respect to the Dirac measure \(\delta(t_k)\).

Hence in claim 1.1 we should have said contact percussion instead of impulsive percussion. The term impulsive force is also currently used to mean contact percussion. Some basic facts about measures are recalled in appendix B. In particular see definition B.3. Although the use of such a vocabulary may appear too complex and too mathematical, we shall see in chapters 2 and 5, section 5.3, that shock dynamics can be formulated as equalities of measures, see (5.32) and (5.34). Then a correct understanding of the different terms that appear in such measure dynamical equations requires to make such distinction between forces and contact percussions.

It is note worthy that the reasoning based on the use of Dirac measures leads to the same algebraic equations of impact dynamics than the classical reasonings done in many mechanical engineering studies, where one considers integrals of the dynamical equations on intervals that tend towards zero. We simply argue that these problems have to be treated with mathematical tools developed by L. Schwartz for researchers working in the field of physics in [478]. Notice that this was advocated

\(^2\)For the moment, by impulsive percussion we mean something like \(p_0\delta(t_k)\), for some real \(p_0\) and some \(t_k\). More precise definitions are given in definitions 1.1 and 1.2.

\(^3\)Note that in the following definition, we do not pretend to define the very basic notion of what a force is. We just set what is meant by a "regular" force, in opposition to an "impulsive" force. For a discussion on the basic definition of what forces are, see for instance [544] and references therein, who argue that in fact, forces in physics should be defined from basic axioms, just like real numbers are in mathematics.
1.1. EXTERNAL PERCUSSIONS

e.g. in [273], but they did not present such a treatment of dynamical systems with impulsive perturbations. In the sequel we shall investigate still other paths using suitable coordinate transformations. Finally let us note that the perfect rigidity hypothesis does not imply point contact (see chapter 6, section 6.4) but a zero measure impact time interval.

Remark 1.1 The above analysis could also be interpreted saying that the Lebesgue measure $dt$ and any atomic measure like $\delta_t$ are mutually singular, see appendix B. This is used for instance in [298] [381] to split the dynamics into smooth and nonsmooth parts as we have done above. These authors use the fact that a function of bounded variation on a closed interval can be uniquely decomposed into the sum of an absolutely continuous (\textsuperscript{4}), a continuous function and a so-called "jump-function", or singular part [394]. We shall come back later in chapter 5 on these works.

Remark 1.2 The solutions of differential equations with distributions can also be studied by considering sequences of equations whose coefficients are functions that tend towards the distributional coefficients [154]. For instance (1.2) is the limit of $\tilde{x}_n = p_n(t)$, where $\{p_n\}$ is a sequence approximating the Dirac measure, see appendix A. Another point of view is to give the approximating problems an \emph{a priori} physical meaning by considering state-dependent forces $\psi_n(x_n)$ that possess certain properties; one has to prove that the limit (with respect to a certain notion of convergence) problem is a dynamical problem involving singular measures [93] [84] [85] [427] [428]. The available studies are however in general restricted to \emph{penalizing} [85] functions $\psi_n$ which are velocity independent, i.e. the impacts are energy-lossless. Recently velocity dependent penalizing functions have been considered in the seminal paper [416]. These works are presented in chapter 2 and chapter 3. An open issue is to prove that we can associate to any rigid problem $P$ with any energetical behaviour at impacts a compliant problem $P_n$, and that any passive or strictly passive \textsuperscript{5} compliant environment is a possible candidate for approximation. We shall come back later on this when we study the relationships between continuous dynamics models and infinitely rigid models in chapter 2.

Example 1.2 Consider a system composed of two masses $m_1$ and $m_2$ moving on a horizontal line, with coordinates $x_1$ and $x_2$, linked with a spring of stiffness $k$ (see figure 1.2). We assume that with a suitable coordinate $x_2$ transformation the spring is at rest when $x_1 = x_2$. $u$ is the (bounded) force applied on mass 1. The dynamical equations of the system are given by:

\begin{align*}
  m_1 \ddot{x}_1 + k(x_1 - x_2) &= u \\
  m_2 \ddot{x}_2 + k(x_2 - x_1) &= p \delta_0
\end{align*}  

\textsuperscript{4}Recall that an absolutely continuous function $f$ is a function that is the indefinite integral of a locally Lebesgue integrable function $h$, i.e. $f(t) = \int h(s)\,ds = \int_a^t h(s)\,ds = \int h\,d\lambda$, for some $a \in \mathbb{R}$, where $\lambda$ is the Lebesgue's measure, also denoted as $ds$, see appendix B. Then $f = h$ $\lambda$-almost-everywhere.

\textsuperscript{5}In the control theory sense, see appendix E.
where we assume that the percussion on mass 2 occurs at $t = 0$. Suppose that $x_1$, $\dot{x}_1$, $x_2$, $\dot{x}_2$ possess a discontinuity at $t = 0$; we thus obtain from (1.6):

\[ m_1\{\ddot{x}_1\} + k(x_1 - x_2) = u \]
\[ m_1\sigma_{x_1}\delta_0 + m_1\sigma_{x_1}\dot{\delta}_0 = 0 \]
\[ m_2\{\ddot{x}_2\} + k(x_2 - x_1) = 0 \]
\[ m_2\sigma_{x_2}\delta_0 + m_2\sigma_{x_2}\dot{\delta}_0 = p\delta_0 \]

From the analysis done in example 1.1 we deduce that $x_1$, $\dot{x}_1$, $x_2$ are continuous time functions whereas $p = m_2\sigma_{x_2}$. Let us denote $x_c(t) = \frac{m_1x_1 + m_2(x_2 - l)}{m_1 + m_2}$ the position of the center of mass of the system, where $l$ is the spring length. Since $\dot{x}_1$ is continuous we have:

\[ \dot{x}_c(0^-) = \frac{m_1\dot{x}_1(0^-) + m_2\dot{x}_2(0^-)}{m_1 + m_2} \]
\[ \dot{x}_c(0^+) = \frac{m_1\dot{x}_1(0^+) + m_2\dot{x}_2(0^+)}{m_1 + m_2} \]

Thus $m_2\sigma_{x_2} = (m_1 + m_2)(\dot{x}_c(0^+) - \dot{x}_c(0^-))$. Also from (1.6) and (1.8) we get:

\[ (m_1 + m_2)\{\dot{x}_c\} = p\delta_0 - (m_1 + m_2)\sigma_{x_2}\dot{\delta}_0 + u \]

which governs the system's center of gravity motion. If there is no loss of energy during the impact, we must have $T(0^+) = \frac{1}{2}m_1\dot{x}_1(0)^2 + m_2\dot{x}_2(0)^2 = T(0^-) = \frac{1}{2}m_1\dot{x}_1(0)^2 + m_2\dot{x}_2(0)^2$, where $T(t)$ denotes the kinetic energy of the system (Note that since potential energy depends on position only, it does not change at the percussion instant). Thus we get $\dot{x}_2(0^+) = -\dot{x}_2(0^-)$ and $p = 2m_2\dot{x}_2(0^+)$ (the other solution leads to an unfeasible motion). One sees that whatever $u$ may be at the moment of impact, the motion of mass 2 is independent of $u$. It can be easily verified that the same result holds (discontinuous $\dot{x}_2$) if a damper is added between the two masses.

**Example 1.3** These ideas can be extended to the case of more complicated mechanical systems such as rigid or flexible joint manipulators. For instance, a rigid manipulator with generalized coordinates vector $q \in \mathbb{R}^n$, submitted to a Cartesian wrench $\lambda \in \mathbb{R}^m, m \leq n$, admits the following state space representation [508]:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -M^{-1}(x_1) \left[ C(x_1, x_2)x_2 + g(x_1) - u - J^T(x_1)\lambda \right]
\end{align*}
\]
where \( x_1 = q, x_2 = \dot{q}, M(x_1) \) is the positive definite inertia matrix, \( C(x_1, x_2) x_2 \) contains centrifugal and Coriolis terms, \( g(x_1) \) is the generalized gravity vector, \( J(x_1) \) is the Jacobian between the joint coordinates \( q \) space and the Cartesian space, i.e. if \( \lambda \) works on \( X \in \mathbb{R}^m \), then \( \dot{X} = J(q) \dot{q} \). We assume that \( J(x_1) \) is full-rank; \( u \) is the generalized torque vector applied at the joints, that can be considered as a control input for the robot manipulator [508], and is assumed to be a bounded function of \( t, x_1, x_2 \).

Proceeding as above, we see that if the system is submitted to a percussion \( \lambda = p_t \delta_{t_k} \) we can write:

\[
\begin{cases}
M(x_1) \{ \dot{x}_2 \} + C(x_1, \{ x_2 \}) \{ x_2 \} + g(x_1) = u \\
M(x_1) \left[ \sigma_q \delta_{t_k} + \sigma_q \delta_{t_k} \right] + 2C(x_1, \{ x_2 \}) \sigma_q \delta_{t_k} \\
+ C(x_1, \sigma_q \delta_{t_k}) \sigma_q \delta_{t_k} = J^T(x_1) p_t \delta_{t_k}
\end{cases}
\tag{1.11}
\]

where we have proceeded as in the foregoing examples to express the distributional derivatives of \( x_2 \) and \( x_1 \), and we have used the properties of \( C(\cdot, \cdot) \), i.e. \( C(x, y)z = C(x, z)y \) and \( C(x, y + z)w = C(x, y)w + C(x, z)w \): the first equation in (1.11) is an equality of functions, whereas the second one is a relation between distributions. Following the reasoning in example 1.1, we deduce from (1.11) that \( \sigma_q(t_k) = q(t_k^+) - q(t_k^-) = 0 \). Moreover note that the last term of the left-hand-side of the second equality is not defined, as the product of two distributions does not exist in general: in this particular case obviously \( \delta_{t_k} \delta_{t_k} \) has absolutely no meaning [478] p.117 [12] §12.5. So in order to render the second equation meaningful, we must have \( \sigma_q(t_k) = 0 \). This proves also that \( J^T(x_1) p_t \delta_{t_k} \) is well defined. Then it yields

\[
M(q) \sigma_q = J^T(q) p_t
\tag{1.12}
\]

From (1.12) we deduce \( p_t = (J(q) J^T(q))^{-1} J(q) M(q) \sigma_q \). Thus claim 1.1 is true for general mechanical systems as well. These systems do not fit within the systems studied e.g. in [298] or [477], since the singular measure is premultiplied by a state dependent term.

### 1.2 Measure differential equations

#### 1.2.1 Some properties

Until now we have considered simple measure differential equations modeling some mechanical systems. Before going on in the next sections with impact dynamics, it is useful to have some more insight on measure differential equations, and which main

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6This is not to be confused with the convolution product of two distributions which in this case is well defined [478].
similarities and discrepancies exist between ordinary differential equations (ODE's) and measure differential equations (MDE's), as well as between MDE's and dynamics of systems with unilateral constraints. The material in this section is taken from Schmaedeke [477]. Let \( S \) be a domain in the \((t, x)\) space \( \mathbb{R}^{n+1} \), \( f(t, x) \) be a real \( n \)-vector function defined on \( S \). Let \( u(t) \) be a real \( m \)-vector function of bounded variation (see appendix C), continuous from the right on a time interval \( I_1 \), and let \( G(t) \) be a continuous \( n \times m \) matrix defined on \( I_1 \). Let \((t_0, x_0)\) be a point in \( S \) with \( t_0 \in I_1 \). Let us denote by \( M \) the differential equation

\[
Dx = f(t, x) + G(t)Du
x(t_0) = x_0
\]  

(1.13)

As we noted in section 1.1, \( D \) denotes the operation of differentiation in the sense of distribution derivatives, with respect to \( t \). A solution \( x(t) \) to \( M \) in (1.13) is defined as follows

**Definition 1.3** A solution \( x(t) \) of \( M \) is a real bounded variation \( n \)-vector together with an interval \( I \) containing \( t_0 \), such that \( x(t) \) is continuous from the right on \( I \) and

- \((t, x(t)) \in S \) for \( t \in I \)
- \( x(t_0) = x_0 \)
- The distributional derivative of \( x(t) \) on \( I \) is \( f(t, x) + G(t)Du \)

Consider now the integral equation \( T \)

\[
x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds + \int_{[t_0, t]} G(s)du(s)
\]  

(1.14)

where \( du \) denotes the Stieltjes measure determined by \( u(t) \). Then we have

**Definition 1.4** A solution \( x(t) \) of \( T \) is a real bounded variation \( n \)-vector \( x(t) \) together with an interval \( I \) such that

- \((t, x(t)) \in S \) for \( t \in I \)
- \( x(t) \) satisfies the integral equation

Then the following theorem generalizes a well-known result for Carathéodory ODE's [200] [554] according to which the solution of an ODE \( \dot{x} = f(t, x), \ x(t_0) = x_0 \), is also a solution of the integral equation \( x(t) = x_0 + \int_{t_0}^{t} f(y, x(y))dy, \) and *vice-versa*.

**Theorem 1.1 ([477])** A solution \( x(t) \) of \( T \) is a solution \( x(t) \) of \( M \) and conversely
1.2. MEASURE DIFFERENTIAL EQUATIONS

Remark 1.3 Notice that if \( u(t) \) is absolutely continuous, then all these definitions reduce to the classical theory for Carathéodory differential equations. In fact in this case the distributional derivative is just the usual derivative. Moreover, the solutions cannot be looked for in other spaces than those of functions of bounded variation. This is because the right-hand-side of \( M \) is a measure, and consequently can only be defined for a nonlinear \( f(\cdot, x) \) if \( x(t) \) is a function. Indeed in general the product of two measures (or distributions) is not a measure (or a distribution).

Following theorem 1.1, the state of the simple system in example 1.1 \( X(t) \) is a solution of (1.2) if and only if it is a solution of the integral equation

\[
X(t) = X(t_0) + \int_{[t_0, t]} AX(\tau)d\tau + \int_{[t_0, t]} \left( \begin{array}{c}
0 \\
Dh(\tau)
\end{array} \right)
\]

and \( X \) (hence \( \dot{x} \)) is necessarily continuous from the right with \( h \) defined as above (We could have also defined \( h \) as a leftcontinuous function since Stieltjes measures can indifferently be defined from functions left as well as rightcontinuous [178] p.133). For the case of a system with a unilateral constraint, e.g. \( x \leq 0 \), it is then possible to initialize the system on the constraint surface, by setting \( \dot{x}(0^-) \) or \( \dot{x}(0^+) \) (provided of course the initial conditions are admissible, i.e. \( \dot{x}(0^+) \leq 0 \) when \( x(0) = 0 \), see e.g. [93]) depending on whether \( h \) is left or rightcontinuous respectively [154] §2, chapter 1. Some authors [381] simply adopt the convention that the left limit \( \dot{x}(0^-) \), which in a sense contains past dynamics, is understood as equal to \( \dot{x}(0) \). Thus clearly a jump is allowed initially, and this also clearly appears in the approach using descriptor variable systems [110] that we shall comment later. This remains true if \( h \) in example 1.1 (see also subsection 1.4.1) is replaced by any function of bounded variation, a useful fact since in certain cases the impacts are likely to occur infinitely often in a finite time interval. Note that right-continuous solutions \( X(t) \) of local bounded variation (RCLBV) are a natural mathematical setting in nonsmooth mechanics of impacts [383] [385], and we shall retrieve this fact when we deal with variational calculus. Moreover we shall use it when we deal with control of impacting systems.

Remark 1.4 Notice however that given a mechanical problem with rigid obstacles, one must rigorously prove that the solution lies effectively in the space of functions of bounded variation: this is an existence result to be proved (it has been attacked and solved in several different places, see e.g. the mathematical studies that we describe in chapter 2 and subsection 5.3.3). Here lies one of the major discrepancies with the theory of measure differential equations, in which the impulsive term is considered a priori. In some simple cases this may be proved easily (for instance by integrating the equations). But in more general cases this can be a difficult problem (for instance for a n-degree-of-freedom system evolving in a prescribed domain). It is however useful to have some knowledge about MDE's when dealing with mechanical systems subject to unilateral constraints, despite of the additional difficulties they involve.
CHAPTER 1. DISTRIBUTIONAL MODEL OF IMPACTS

Remark 1.5 Another major discrepancy between a system with unilateral constraints (with an autonomous continuous vector field) and a measure differential equation as in $\mathcal{M}$, is that the former defines an autonomous system, i.e. the solutions verifies the semi-group property $x(t; t_0, x_0) = x(t - t_0; 0, x_0)$ [27] (here we assume that the constraints are associated to a suitable restitution rule, which associates a jump to the velocity each time the constraint is attained). As we shall see in sub section 1.3.2), the total flow of the system is the concatenation of two flows: one for the continuous part, one for the discrete mapping part of the system. On the contrary in $\mathcal{M}$, $u(t)$ is a priori an exogeneous signal. If one can prove that the velocity of a mechanical system subject to a unilateral constraint is of bounded variation then this shows the existence of the function $h(t)$ in example 1.1 or subsection 1.4.1. One can forget about the dependence of $h(t)$ on the state (preimpact velocities) and on the way the impulse is actually created (this is an "external" percussion) and treat it as $u(t)$ to analyze properties of the solutions. But then clearly something is lost, because the system is no longer considered as an autonomous system. Moreover there must exist a step where one is led to consider the restitution rule that relates post and pre-impact velocities. In other words, considering exogeneous impulses may be a useful first step in the study of impacting systems, but is far from being sufficient.

Theorems on existence and uniqueness of solutions are also generalized to measure differential equations. Ordinary Carathéodory differential equations are generalized to Carathéodory Measure Systems (CMS) defined as follows

Definition 1.5 ([477]) Consider a MDE as in $\mathcal{M}$. Let $f(t, x)$ be defined in a neighborhood of a domain $S$ of $\mathbb{R}^{n+1}$ such that for each point $(t_0, x_0) \in S$ there is a rectangle $R_{ab}$ centered at $(t_0, x_0)$, a constant $K > 0$ and a function $\tau(t)$ summable on the interval $[t_0 - a, t_0 + a]$, such that:

- $f(t, x)$ is measurable in $t$ for each fixed $x$ such that $(t, x) \in R_{ab}$
- $f(t, x)$ is locally Lipschitz continuous with constant $K$ with respect to $x$, for all $(t, x) \in R_{ab}$
- $|f(t, x)| \leq \tau(t)$ in $R_{ab}$
- $\|\int_{t_0}^{t} G(s)du\|^* < b$, where the norm is taken on $[t_0 - a, t_0 + a]$ \(^7\).

It can be checked that $\mathcal{M}$ with $G(t)$ continuous on an interval $I$, $u(t)$ right-continuous of bounded variation on $I$, $[t_0 - a, t_0 + a] \subset I$, is a CMS. What is remarkable in definition 1.5 is that the initial conditions are taken in a domain $S$, but one needs to consider the dynamics outside $S$ (Indeed the rectangles $R_{ab}$ need not be contained in $S$). This is not the case for ordinary Carathéodory equations.

\(^7\)The norm $\|.|\|^*$ is defined in appendix C.
1.2. MEASURE DIFFERENTIAL EQUATIONS

(see subsection 1.4.1). This is intuitively explained by the fact that for any bounded domain within which the initial conditions may lie, then the jump imposed on the state by \( Du(t) \) is likely to take the state outside this domain (instantaneously). This jump \( G(t_k)\sigma_{u(t_k)} \) is clearly independent of the initial conditions: it is an exogeneous variable given by \( u(t) \) and \( G(t) \). Theorem 1.1 is useful to prove local existence and uniqueness results for measure differential equations as \( M \), using the fixed point property of contraction mappings, see [477] theorems 2 and 3, quite similarly as for ODE’s [554].

The following theorem is an extension of the global existence and uniqueness results for ODE’s to MDE’s:

**Theorem 1.2 ([477])** Consider the MDE in \( M \), satisfying the conditions in definition 1.5. Then there exists a unique solution \( \varphi(t, t_0, x_0) \) of \( M \) for every point \( (t_0, x_0) \in S \), where \( \varphi(t, t_0, x_0) \) is defined on a maximal open interval \( (a, b) \ni t_0 \).

It can be easily checked that the systems we deal with in example 1.1 and example 1.2 are Caratheodory measure systems as long as we consider exogeneous impulsive forces of the form \( \sum_{k \geq 0} p_k \delta_{t_k} \) with \( \sum_{k \geq 0} |p_k| < +\infty \), and that theorem 1.2 applies. The classical results for ODE’s concerning maximal extension of the solution, global existence when \( f(t, x) \) is Lipschitz continuous also extend to CMS. The result according to which a maximal solution defined on an interval \( (a, b) \) with \( b < +\infty \) leaves any compact set of the domain of definition of \( f(t, x) \) (either it tends to the boundary of that domain, or it escapes, or both [200] §5), is generalized to CMS [477] theorems 5, 6 and corollaries 1, 2.

Let us note also from [477] theorem 9 that the solution of a measure differential equation is generally not continuous with respect to initial time \( t_0 \) but of bounded variation in \( t_0 \). Example (4.5) in [477] analyzes the first order equation \( Dx = x + \delta_0 \) where the solutions \( \varphi_1(t, t_1, 1) \) starting at \( t_1 < 0, x(t_1) = 1 \), and \( \varphi_2(t, t_2, 1) \) starting at \( t_2 > 0, x(t_2) = 1 \), are given by:

- For \( t \in [t_1, 0) \), \( \varphi(t; t_1, 1) = \exp(t - t_1) \), and for \( t \in (0, +\infty) \), \( \varphi(t; t_1, 1) = (\exp(-t_1) + 1) \exp(t) \)
- On \( [t_2, +\infty) \), \( \varphi(t; t_2, 1) = \exp(t - t_2) \), that is clearly not affected by the Dirac measure.

Those solutions are such that

\[
\varphi_1(t_2, t_1, 1) - \varphi_2(t_2, t_2, 1) = \exp(t_2)\{\exp(-t_1) + 1\} - 1 \tag{1.16}
\]

that is close to 1 when \( t_1 \) and \( t_2 \) are close to zero. Clearly \( \varphi(0, \cdot, 1) \) is discontinuous at \( t_0 = 0 \). For \( \varphi(0, \cdot, 1) \) to be continuous at \( t_0 = 0 \) would require that \( \varphi(0, t_1, 1) - \varphi(0, t_2, 1) \to 0 \) as \( t_1 \to 0 \) and \( t_2 \to 0 \). However if \( t_1 < 0 < t_2 \), and since we specify that solutions are right-continuous, we have \( \varphi(0, t_1, 1) = \varphi(0^+, t_1, 1) = \exp(-t_1) + 1 \) while \( \varphi(0, t_2, 1) = \exp(-t_2) \), so that one gets:

\[
\varphi(0, t_1, 1) - \varphi(0, t_2, 1) \to 1 \tag{1.17}
\]
12 CHAPTER 1. DISTRIBUTIONAL MODEL OF IMPACTS

It is worth remarking the big difference between simple Carathéodory differential equations, for which uniqueness and continuous dependence on initial data \((t_0, x_0)\) are both guaranteed by the local Lipschitz continuity of the vector field [154] chapter 1, [95] §1.10, chapter 2, and measure differential equations for which this is not true.

However, continuity with respect to \(x_0\) is true ([477] theorem 9) and is easily proved following classical arguments for ODE’s (see e.g. [554]). Indeed, let us assume the existence of solutions of \(\mathcal{M}\) in (1.13) on an interval \((a, b)\). Let us consider \(x_0 = x(t_0)\) and \(y_0 = y(t_0)\), \(t_0 \in (a, b)\). Then by (1.14) we have

\[
\varphi(t; t_0, x_0) - \varphi(t; t_0, y_0) = x_0 - y_0 + \int_{t_0}^{t} \{f(s, x(s)) - f(s, y(s))\} \, ds
\]

where \(x(t) = \varphi(t; t_0, x_0)\) and \(y(t) = \varphi(t; t_0, y_0)\). Hence the proof of theorem 2.4.57 in [554] applies directly, and the solutions of \(\mathcal{M}\) depend continuously on the initial state condition, on any finite time interval.

**Remark 1.6** Our main topic is mechanical systems subject to rigid body impacts. We shall see in subsection 1.3 that they are represented by some kind of measure differential equations. Let us note that there are still other types of MDE’s that have been studied in the literature. For instance, Bainov and Simeonov [27] study systems with impulsive effects that can be described as

\[
\begin{align*}
\dot{x} &= f(x, t), \quad t \neq t_k(x) \\
\sigma_x(t_k) &= I_k(x(t_k^+)), \quad t = t_k(x)
\end{align*}
\]

Some assumptions on the discontinuities times \(t_k\) are made in [27], that we shall describe later in chapter 7. Results on continuous dependence of solutions of such systems with respect to initial data and parameters can be found in [27]. For the sake of brevity of the exposition we do not reproduce these results here. Let us simply note that even if the \(t_k\)’s in (1.19) are fixed, such systems are different from the MDE’s in (1.13) since the jumps times are state-dependent. As an illustration, let us consider the system

\[
\begin{align*}
\dot{x} &= 0 \quad \text{for } t \neq t_k \\
\sigma_x(t_k) &= 1 \\
t_k(x) &= \{t : t = x(t) + 2\}
\end{align*}
\]

Solutions are given by \(\varphi(t; t_0, x_0) = x_0\), and the first discontinuity occurs at \(t_0 = \varphi(t_0; t_0, x_0) + 2 = x_0 + 2\). Hence the solution jumps to \(x_0 + 1 = \varphi(t_0^+; t_0, x_0)\). Then a second jump occurs at \(t_1 = \varphi(t_1; t_0^+, x(t_0^+)) + 2, \text{ i.e. } t_1 = x_0 + 3\), with \(x(t_1^+) = x_0 + 2\), and so on.
1.2. MEASURE DIFFERENTIAL EQUATIONS

1.2.2 Additional comments

It is not our goal here to describe in detail the theory of distributions, and all (difficult) problems related to existence, uniqueness and stability of differential equations with distributions in coefficients. In this book we are concerned with shock dynamics, for which Schwartz' distributions or measure theory are sufficient. We however encountered some mathematical objects which are not properly defined using these tools. Is it possible to use other paths to provide a meaning to differential equations containing such objects? The answer is yes in certain cases. We briefly review this now. Let us note first that employing the technique mentioned in section 1.1, i.e. separation of regular distributions (functions) and singular distributions (Dirac measures and their derivatives), allows us to treat quite simply second order differential equations with impulsive perturbations like the ones in this note. Notice that this is not always possible: indeed consider the differential equation

\[ \dot{x} = f(x) + g(x)\delta_{t_k} \quad (1.21) \]

where \( x \in \mathbb{R}, f, g \) smooth functions of \( x \). Then we should get \( \{\dot{x}\} + \sigma_x(t_k)\delta_{t_k} = f(x) + g(x)\delta_{t_k} \), so that

- \( \{\dot{x}\} = f(x) \)
- \( \sigma_x(t_k) = g(x) \).

But the second equality is meaningless: indeed the term \( g(x)\delta_{t_k} \) represents in fact a distribution, i.e. for any function \( \phi(t) \) with support \( K_\phi \) containing \( t_k \) and continuous at \( t_k \), \( \langle g(x)\delta_{t_k}, \phi \rangle = \int_{K_\phi} g(x)\delta_{t_k} \phi(t) dt = g(x(t_k))\phi(t_k) \). Thus we should write the second equality as \( \sigma_x(t_k) = g(x(t_k)) \). But \( x(t_k) \) is not well-defined and in general \( g(x(t_k)) \) is neither (Of course if \( g(x) \) is replaced by a function of time \( g(t) \) then the technique can be employed provided \( g(t) \) is continuous at \( t_k \), see [154] pp.40-41. This is what is described in subsection 1.2). Such problems are treated e.g. in [297] [154]: such differential equations do not have in general a unique solution (independently of the choice of the initial data), and the obtained solution strongly depends on the sequence of problems considered to approximate the equation [297] theorems 5.1 and 5.2, [154] theorem 4, chapter 1, §3. Kurzweil develops a theoretical framework on a class of differential equations (Kurzweil Differential Equations, KDE) in [297], see also [19] §2,3,4 for more accessible explanations on KDE's and Kurzweil integrals. Roughly speaking, the underlying idea is to consider ODE's whose right-hand-side may not converge to a differentiable function, or even not to a function (like when delta-sequences are considered). Then one rather looks at the associated integral equation (that is a KDE), just forgetting about the differential formulation that might not be defined. Kurzweil provides conditions such that the integral has a meaning, by defining a Kurzweil integral from Kurzweil-Riemann sums. One may construct a KDE from any ODE that satisfies Carthéodory conditions. Then the solution of the KDE and that of the ODE are equal. In this sense KDE's really represent an extension of ODE's. Theorems 5.1 and 5.2 in [297] provide existential
results for MDE's as in (1.21), with solutions of bounded variation. It is shown that a solution exists outside the critical point $t_k$, in the sense that the solutions of an approximating sequence of ODE's (where $\delta_{t_k}$ is replaced by a delta-sequence) converge uniformly outside $t_k$.

Related more recent work can be found in [66] where the author studies existence and uniqueness of solutions of equations like

$$\dot{x} = f(x) + g(x)u$$

\(x \in \mathbb{R}^n, u \text{ scalar of bounded variation}^8\).

In relation with these considerations, recall that controllability of linear systems may be analyzed via impulsive inputs (combinations of $\delta_t, \dot{\delta}_t$ ... ) [252] section 2.3. The system is then controllable if there exists such an input that produces an arbitrary jump in the state vector. In case of nonlinear systems, it seems in view of these discussions that such an analysis is much less simple since in general solutions will not be unique, or uniqueness can be obtained at the price of interpreting differently the notion of a solution. In case of mechanical systems like in example 1.3, we have seen that a discontinuous position yields some non-wellposed problem when one uses distributions theory. Thus a practical viewpoint (impacts are created using some hammer-like device) of impact dynamics yields to consider Dirac measures only as possible impulsive inputs. Such impulsive controllers have been proven to be useful in some cases [15] [593] for systems with static dry friction. This is also related to the problem of activation functions studied in [366] (existence of solutions to the sweeping process that we describe in chapter 5 is studied in [366]), that may represent some pinball games.

Although the differential equations we shall deal with will not fall into the above category (see (1.22)), one should note that the work of the impulsive force at the impact time is given by an expression similar to $W \overset{\Delta}{=} \int_{t_0}^{t_1} \dot{q}(t)\delta_{t_k} dt$, with $t_k \in [t_0, t_1]$. Since $\dot{q}$ is discontinuous at $t_k$, the integrand is meaningless, and this is exactly the same kind of problems as in the above differential equations. This reveals a limitation of the theory of distributions in the impact dynamics modeling. We shall discuss about this in chapter 2 when we study continuous-dynamics models as approximating models for the perfect rigid case (see also chapter 7, subsection 7.1.3). Concerning the above $W$, some questions arise:

- Although the integrand has no meaning (i.e. is not properly defined) if we rely on Schwartz' distributions theory, can it be given a sense via a Kurzweil integral?

\(^8\)It is noteworthy that for linear systems, there always exists a global generalized change of coordinates in $x$ and $u$ such that the derivatives of the input disappear in the new coordinate system, see remark 1.12. This is not the case for nonlinear systems, hence the interest of considering such equations. According to [70] lemma 8.1, such a transformation locally exists for such systems linear in $u$. 

1.2. MEASURE DIFFERENTIAL EQUATIONS

- If \( q_n(t) \) is the solution of an approximating problem where \( \delta_{t_k} \) is replaced by any delta-sequence (see appendix A, section A.2), can we prove that \( W_n \) converges to a real \( W \)?

- Intuitively one deduces that \( W \) should equal the loss of kinetic energy at impact. Can this be shown rigorously?

We shall provide a partial answer to the second point in chapter 2, section 6.3.

Bressan's hyperimpulsive systems

Systems for which both the position and the velocity possess discontinuities have been identified and studied by Bressan in [70]. They have been given the name of hyperimpulsive. Such systems arise when one considers some of the coordinates as control variables (An example is a man on a swing, where the length of the swing can be varied to produce a desired movement. The angle of the swing with the vertical can also be considered as a control [68]. Other applications are control of skis or of a single rigid body [454]). Deep mathematical investigations are needed to study such systems, related to nature and existence of solutions. As long as generalized forces are control variables, this does not occur.

The goal in hyperimpulsive systems is to find out generalized coordinates \( \gamma_1, \ldots, \gamma_m \) such that the Hamiltonian system \( \dot{q} = Q(q, p, \gamma_1, \gamma_2), \dot{p} = P(q, p, \gamma_1, \gamma_2) \) can be controlled via the \( \gamma_i \)'s. Thus choosing a discontinuous \( \gamma_i \) introduces an hyperimpulsive term in the dynamics. In general the \( \gamma_i \)'s appear quadratically in the dynamical equations. Assume that \( m = 1 \). The general form of the Lagrange dynamical equations is

\[
\dot{x} = f(t, x, \gamma) + g(t, x, \gamma)\dot{\gamma} + h(t, x, \gamma)\dot{\gamma}^2 \tag{1.23}
\]

There are two difficulties in the analysis of such systems when \( \gamma(t) \) possesses discontinuities: first the second term in the right-hand-side make it similar to the equation in (1.22), second \( \dot{\gamma}(t) \) is a Dirac distribution at the discontinuity times, so that its square has no meaning in the theory of distributions, see appendix A.

Therefore the main problem may be to find conditions under which some of the derivatives appear only linearly (i.e. \( h(t, x, \gamma) = 0 \)), and are thus controlizable [454]. Bressan [66] does not choose the distributional approach and overcomes the first problem by considering properties of the input/state map \( \Phi(u) \), and shows that if \( \Phi \) is Lipschitz continuous with respect to a suitable \( L_1 \) norm, then a unique (up to \( L_1 \)-equivalence) generalized solution (in the sense defined in [66]) exists, provided \( \|\Phi_u\|_{L_1} \) is bounded in \( R^n \). It is seen on an example that the approach has the effect of "smoothing" the solution, i.e. the jumps at isolated points may be ignored. The work is extended in [67] when \( u \) is a vector; roughly speaking, discontinuous functions (\( u \) and the solution) are replaced by their graph-completion. The practical importance of having a function \( \Phi \) with good properties is for robustness purposes.
on possible uncertainties in the control \([454]\). Other related studies on the topic when \(h(t, x, \gamma) = 0\) can be found in \([119] [455] [472]\).

The case where \(h(t, x, \gamma) \neq 0\) has received attention in \([68]\). Then in addition to the difficulty due to the second term in the right-hand-side of (1.23), the third term has to be analyzed since it involves the square of the derivative of a possibly discontinuous function. As we have seen in example 1.3, if one relies on Schwartz’ distributions theory, the product \(\delta_{x_k}\delta_t\) is not defined (there are very particular cases in which the square of the Dirac distribution can be given a sense, see appendix A). To overcome all those obstacles, Bressan and Rampazzo \([68]\) prove that the trajectories of system 1.23 with discontinuous \(\gamma\), when \(\gamma\) is the limit of a sequence \(\{\gamma_n\}\) of functions \(\gamma_n \in W^{1,2}(I)\) \(^9\) can be given a sense. More precisely, the system in (1.23) is equivalent to an affine in the input system of the form:

\[
\dot{y} = f(y) + g(y)v + h(\gamma)\tilde{w},
\]

Equivalence is understood in the sense that any sequence \(\{x_n\}\) of solutions of (1.23) corresponding to a sequence of inputs \(\{\gamma_n\}\) with \(\gamma_n \in W^{1,2}(I)\) and \(\|\gamma_n\|_{L^2} \leq \sqrt{K}\) (i.e. \(\gamma_n \in \Gamma_K\)), converges in \(L^1(I)\) to some solution of (1.24) with \(v \in L^2(I)\), \(\|v\|_{L^2} \leq \sqrt{K}\), \(w\) non-decreasing, and \(\int_a^b v^2(\tau)\,d\tau \leq w(b) - w(a)\). Conversely, to any solution \(y\) of (1.24), one is able to associate a sequence of solutions \(\{x_n\}\) of (1.23) corresponding to a sequence of inputs \(\{\gamma_n\}\), \(\gamma_n \in \Gamma_K\), that converges towards \(y\) in \(L^1(I)\) (see \([68]\) theorems 3.1 and 3.2). The restriction \(\|\gamma_n\|_{L^2} \leq \sqrt{K}\) for some \(0 \leq K < +\infty\) is shown to be essential, and corresponds in mechanical systems to a bound on the kinetic energy. It is also shown in \([68]\) that the set of attainable points at \(t = \tau\) in the state space of system (1.24), with controllers \(v\) and \(w\) as above (or with \(\gamma \in \Gamma_K\) with \(\gamma\) in a compact set \(C_\gamma\)), is compact. This enables one to assure the existence of an optimal controller for the variational problem

\[
\min\{\Phi(x(\tau)), x\ \text{solution of (1.23)}\} \quad (1.25)
\]

when \(\Phi\) is lower semicontinuous (see appendix D, see also Tonelli’s theorem 3.1 in chapter 3. An example of mechanical system with displacement jumps is also given in \([27]\) (see example 6.2, p.80), but the dynamics are such that \(g(t, x, \gamma) = h(t, x, \gamma) = 0\) (this is a pendulum whose point of suspension is forced to have a discontinuous vertical motion, and the dynamics are linearized around the vertical position). In \([68]\) the angular displacement is taken as the -possibly- discontinuous controller and the length of the pendulum is variable. This introduces quadratic terms in the dynamics which justify the general expression as in (1.23).

Note that displacement discontinuities are also very common phenomena in models of mechanical systems having non-monotone load-displacement equilibrium paths. Roughly, the decreasing portion of such load-displacement equilibrium path is dynamically unstable, and the system deviates dynamically from such equilibrium heading towards a stable non-adjacent equilibrium state. Such problems are

\(^9\)This is a Sobolev space, see appendix C.
related to existence of classical solutions in some dynamical systems subject to unilateral constraints and Coulomb friction \(^{(10)}\). An example is given in \([282]\) of a two-degree-of-freedom system (a particle sliding on a rigid surface with Coulomb friction, and acted upon by springs and dampers), for which there exist initial data and stiffness values compatible with the system's constraints such that it is not possible to integrate the dynamics in the future time. One interesting question is to provide a physical meaning to such situations, i.e. being able to characterize such behaviours as the limit of well-behaved dynamical processes (for instance when one admits displacement jumps in a quasistatic problem, are those discontinuous solutions consistent with the fundamental principle of dynamics\(^?\)). In \([343]\) a generalized quasistatic problem is formulated which allows for displacement discontinuities in the "quasistatic" solution (the constrained particle is considered massless), provided they lead to energy dissipation into some external sink. More precisely, when damping is absent, there may be situations in which there is no classical solution (i.e. time continuous solutions) to the quasistatic problem. But by adding damping, Martins et al \([343]\) show that there is always at least one classical solution. The underlying idea is then to let the damping coefficient tend towards zero, and analyze what happens in the limit. After a sophisticated analysis, the authors are able to give a physical interpretation to the displacement jumps that occur in certain critical situations. Thus in this quasistatic context the instantaneous path replaces the fast dynamic transition between an unstable equilibrium and a stable non-adjacent equilibrium position. Such displacement jumps can also be approximated \textit{via} sequences of dynamic problems \([344]\), where both the mass and the damping coefficient tend to zero in the limit. The limit in \([343]\) is shown to coincide with that in \([344]\). Most importantly, the limit solution is shown to possess a dissipation property: very roughly, if \(q_\varepsilon\) denotes the solution (motion of the particle) and \(\varepsilon\) the dimensionless coefficient such that the mass is proportional to \(\varepsilon^2\) while damping is proportional to \(\varepsilon\), let 
\[
F \triangleq \lim_{\varepsilon \to 0} \varepsilon^2 q_\varepsilon + 2\varepsilon q_\varepsilon \quad \text{and} \quad Q \triangleq \lim_{\varepsilon \to 0} q_\varepsilon:
\]
then the operator \(Q \mapsto FQ\) is shown to satisfy a passivity property.

Those results are quite important, because we shall see throughout the remaining chapters that an elegant way to give a meaning to apparently non-wellposed dynamical problems is to consider them as the limit of sequences of wellposed problems. Clearly although both hyperimpulsive systems and quasistatic systems with Coulomb friction both involve discontinuous displacements, the underlying mathematical problems are of different nature.

\(^{10}\text{We shall retrieve such problems in chapter 5, sections 5.4 and 5.4.2 when we deal with existence of solutions of complementarity problems. But only velocities discontinuities will then be involved.}\)
1.3 Systems subject to unilateral constraints

1.3.1 General considerations

Definition

Classically, the percussion problem may be attacked via either external impacts or instantaneous imposition of constraints. We have for the moment considered the impacts on the system as purely exogeneous signals having a particular form, namely a sequence of Dirac measures \(^{(11)}\). One may legitimately wonder how such signals may be created on a mechanical structure. This is in fact closely related (at least for mechanical systems) to unilateral constraints. Before going on with the relationships between such constraints and impulses, let us define what is meant by a unilateral constraint.

Definition 1.6 Let a Lagrangian system be described by a set of generalized co-ordinates \(q \in \mathbb{R}^n\), and let \(f(q) = 0\) be a smooth submanifold of codimension \(m < n\) in the configuration space of the system, such that \(\nabla_q f_i(q) \neq 0\) in the region of interest (i.e. the domain within which the system evolves). Then the inequality \(f(q) \geq 0\) defines a subspace of the configuration space, namely \(\Phi = \{q : f(q) \geq 0\} = \cap_{i=1}^m \{q : f_i(q) > 0\}\), where the system is constrained to evolve.

For the definition of a submanifold, we refer the reader to [1] [16] [200]. Roughly, if the ambient space is of dimension \(n\), a submanifold of codimension \(m\) \(^{(12)}\) is simply a surface of dimension \(n-m\) (if \(m = 1\) one speaks of a hypersurface) that is embedded into the theoretical setting of manifolds, i.e. is endowed with a special local coordinates structure (a manifold is a set which has locally the structure of a Euclidean space, with linear coordinates). Smooth submanifolds are those that admit a tangent space at each point \(q\). We impose here that the gradient of \(f(q)\) be different from zero at each point, hence the normal direction to the surface of constraint is well-defined. As we shall see this is important when we consider the generalized interaction forces acting from the constraint on the system. Note that we can also consider time-varying submanifolds \(f(q, t) = 0\), provided \(f(q, t)\) is continuous in \(t\). Indeed it is important that at the collision times, continuity with respect to \(t\) holds.

\(^{(11)}\)This is also the point of view taken for instance in [298] [299] [300].

\(^{(12)}\)The codimension of a submanifold (a surface) is the difference between the dimension of the ambient space and the dimension of the submanifold [16]. Recall that there are 3 manners of defining a surface \(S\) of dimension \(n - m\) in an ambient space of dimension \(n\) [131]. We make use only of one of them, which consists of defining \(S\) through \(m\) relationships like \(f_i(q_1, \ldots, q_n) = 0\). A non-singular point \(q_0\) is such that the matrix \((\frac{\partial f_i}{\partial q_j}(q_0)) \in \mathbb{R}^{m \times m}\) has rank \(m\). Then the three definitions are equivalent in a neighborhood of \(q_0\). The codimension of the intersection \(S_1 \cap S_2\) is the sum of the codimensions of \(S_1\) and \(S_2\), provided the intersection is transversal (i.e. the tangent hyperplanes to each one of the surfaces at the intersection span the whole ambient space [131] p.50. The reader can think of two planes in \(\mathbb{R}^3\) (codimension 1 surfaces): either they are parallel, or they intersect transversally and the intersection is a straight line whose codimension is 2).
1.3. SYSTEMS SUBJECT TO UNILATERAL CONSTRAINTS

Otherwise the normal direction to the constraint surface may have a discontinuity at the same instant as bodies strike, and this poses difficulties for the definition of the system's evolution at the impact, see chapter 6. Although mechanical systems composed of colliding rigid bodies do not involve such constraints, it may be that in some control problems one introduces via feedback some discontinuity in the "closed-loop" constraint, see subsection 7.1.4. It is noteworthy that a constraint \( f(q) = 0 \) does not necessarily correspond to some fixed environment in which the system collides. As we shall see, in general \( f(q) \) may represent a sort of distance between bodies, in the configuration space (hence \( n \)-dimensional).

**Remark 1.7** The unilateral constraints can be in general formulated either as \( f(q) \geq 0 \) or as \( g(q) \leq 0 \). In the first case the normal vector \( \nabla q f(q) \) points outwards the constraint surface. In the second case \( \nabla q g(q) \) points inwards. It is therefore not very important to adopt one or the other convention. However one should be aware of that fact to compute the admissible normal interaction forces or impulsions.

**Loss of linearity**

One of the main consequences of the addition of unilateral constraints on a system, is that even if the free-motion dynamics are linear, i.e.

\[
\dot{x} = Ax + Bu
\]  

(1.26)

the total system with a set of inequalities

\[
Cx \geq D
\]  

(1.27)

define a nonlinear system. Systems as in (1.26) (1.27) are sometimes called convex systems. If \( x(t) = \varphi(t; x_0, u_0) \) and \( y(t) = \varphi(t; x_1, u_1) \) are solutions of the controlled system in (1.26) (1.27), then for all \( \lambda \in [0, 1] \), \( \lambda x(t) + (1 - \lambda) y(t) \) is a solution also (117). They are called convex conical if \( D \equiv 0 \). It is in fact not difficult to understand where the nonlinearity comes from: indeed the solutions of such systems possess (see subsection 1.3.1) discontinuities at certain times \( t_k \). In turn, the \( t_k \)'s are in general nonlinear functions of the initial conditions. Hence the superposition principle for linear systems no longer holds. This may also be seen at once by noting that if such systems were linear, then \( \varphi(t; \lambda x_0, \lambda u_0) \) would be equal to \( \lambda \varphi(t; x_0, u_0) \). Now there is no reason why \( \lambda \varphi(t; x_0, u_0) \) should be a solution of (1.26) plus (1.27) for any \( \lambda \in \mathbb{R} \) (think of the case \( C = I, D = 0 \) and take \( \lambda = -1 \)). We shall retrieve the nonlinearity when we deal with impact Poincaré maps (see chapter 7). Then the main and fundamental discrepancy between the classical discretization of linear or nonlinear systems and the calculation of such Poincaré maps, lies in the fact that the "sampling" times for the latter are (nonlinear) functions of the system's state. This introduces much difficulties in the dynamical analysis. In fact as we shall see, even in apparently very simple cases, it is impossible to calculate explicitly the Poincaré map, whose dynamics may display a very complex behaviour.
Remark 1.8 On the contrary, systems with fixed exogeneous instants of impulse \( \{t_k\} \)\(^{27}\) are linear. For instance, the system

\[
\dot{x} = Ax \quad \text{for } t \neq t_k
\]

\[
\sigma_x(t_k) = B_k \dot{x}(t_k^-) \quad \text{for } t = t_k
\]

(1.28)

is linear. The solution is given by

\[
\varphi(t; \tau_0, x_0) = \exp(A(t - t_k)) \Pi_{j=1}^{i+1} U_j(I + B_i) \exp(A(t_j - \tau_0))x_0
\]

(1.29)

for \( t_{i-1} \leq \tau_0 < t_k < t \leq t_{k+1} \) and some matrix \( U_j \).

Necessity of collisions

The analysis done in example 1.1 is valid each time there is an impulsive force that acts on the system, and the discontinuity in the velocity is from a logical point of view to be seen as a consequence of the impulse. In fact, we have considered the dynamics of the mass as a measure differential equation, without taking care on how the impulsive behaviour could be created. Conversely, if we consider that the mass collides a rigid wall at \( t = t_k, x(t_k) = 0 \) (the unilateral constraint being \( x \leq 0 \) ) then \( \dot{x} \) must be discontinuous at \( t_k \) except if \( \dot{x}(t_k^-) = 0 \), since it must happen that \( \text{sgn}(\dot{x}(t_k^-)) = -\text{sgn}(\dot{x}(t_k^+)) \) (otherwise the mass would penetrate into the constraint, a situation that is precluded by the unilaterality condition). Then the percussion is rather a consequence of the discontinuity. Now why should the position remain continuous at the impact instant? If \( x \) has a discontinuity at \( t_k \), then the contact percussion has the form \( p_1 \delta_{t_k} + p_2 \dot{\delta}_{t_k} \). From a mathematical point of view, nothing \textit{a priori} hampers the interaction force to be like this. Then it means that the dynamical "rigid" problem \( P \) (i.e. the differential equation with distribution in coefficients together with its initial data) can be approximated by a sequence of "compliant" or "continuous-dynamics" problems \( P_n \) where the interaction force has the value \( p_1 \delta_n + p_2 \dot{\delta}_n \)\(^{154}\) chapter 1 §2, where \( \delta_n \) is a sequence of functions with continuous first derivatives that determines the Dirac measure and \( \delta_n \) determines the distribution \( \delta_{t_k} \) (see appendix A.2). However it is really not clear which kind of physical model should correspond to such approximating problems. In chapter 2 we shall prove that for simple (i.e. integrable) continuous-dynamics models \( p_2 \equiv 0 \). Let us note at once that if the position has a discontinuity, then the integrand of the impulsive work contains \( \delta_{t_k} \delta_{t_k} \) which has no meaning, but would represent as the limit of approximating sequences an "object" (not a function, not a distribution) with mass \( +\infty \) concentrated at \( t_k \)\(^{478}\) p.117, see also\(^{111}\) proposition 1 \(^{13}\), see also Bressan and Rampazzo's results \(^{68}\) in subsection 1.2.2 (\(^{14}\)) we conjecture

\(^{13}\)It can be shown \(^{111}\) that for any \( \varphi \in \mathcal{D} \), any sequence of functions \( \delta_n \in L_2 \) converging weakly* towards \( \delta \in \mathcal{D}'^* \), then \( ||\delta_n \varphi||_2 \to +\infty \). It is in some particular cases possible to give a sense to distributions containing \( \delta^2 \), see appendix A

\(^{14}\)But recall that in their work, the impulsive behaviour is supposed to be created via the controller, and is \textit{a priori} not the result of a physical process like a shock.
that physical passive or strictly passive compliant approximating models cannot yield such results in the limit, hence a continuous position. This is confirmed by the studies in chapter 2, sections 6.3 and 2.2. At least the conjecture seems to be true for frictionless constraints.

In the more general setting of a Lagrangian system with a frictionless unilateral constraint $f(q) \geq 0$, then it is clear that a necessary and sufficient condition (in the frictionless case) for the interaction force not to be impulsive is that $\dot{q}^T(t_k)\nabla_q f(q)(t_k) = 0$ at the time when contact is established, i.e. $f(q(t_k)) = 0$. If the generalized velocity points inwards the constraint, then an impact must occur. Such conditions have been for instance set in [381] proposition 2.2. As we shall see in chapter 5, subsection 5.4.2, the case when Coulomb friction is introduced may create certain unexpected behaviours.

Let us now describe some examples of mechanical systems where unilateral constraints appear.

Example 1.4 A typical example of a system submitted to a unilateral constraint is the yo-yo, see figure 1.3. The maximum length of the string is $L$. We suppose that the string is not deformable in the horizontal direction, and we neglect the variation of the radius $r$ when $x(t)$ increases or decreases (i.e. the disk center of gravity $G$ performs a pure vertical motion as long as $x \leq L + r$). Hence there is one unilateral constraint in the dynamics, given by $x \leq L + r$. The free-motion dynamical equations of the yo-yo are given when $x \leq L$ by:

$$
\begin{align*}
I\ddot{\theta} &= -r \dot{R} \\
mx &= mg - R
\end{align*}
$$

with $x = -r\theta$. Hence the dynamics when $x \leq L$ can be written as:

$$
\begin{align*}
(I + mr^2)\ddot{x} &= mgr^2 \\
x &= -r\theta
\end{align*}
$$
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Figure 1.4: There is no shock in the yo-yo.

with \( x(0) = 0, \dot{x}(0) = \dot{x}_0 \). \( y = r \) during this phase of motion due to the above assumption on the string motion. Now when \( L < x \leq L + r \), the disk starts to rotate around the point \( B \) at which the string is attached. Then the dynamics are given by:

\[
\begin{align*}
\dot{x} &= mg - R \\
\dot{\theta} &= -g \cos(\theta - \theta_L)
\end{align*}
\]  

(1.32)

where \( \theta_L \) denotes the value of \( \theta \) when \( x = L \) (i.e. when the disk starts its rotation with \( B \) the instantaneous rotation center; \( B \) is fixed during this phase due to the above assumption). Now we have also kinematic relationships, \( x - L = r \sin(\theta_L - \theta) \) and \( y = r \cos(\theta_L - \theta) \). The disk attains its lower position when \( x = L + r \) and \( y = 0 \), that is when \( \theta = \theta_L = -\frac{\pi}{2} \), at \( t = t_0 \). Then it follows that \( \dot{x}(t_0) = 0 \) and \( \dot{y}(t_0) = r \dot{\theta}(t_0) < 0 \). Consider now the constraint surface \( x = L + r \) in the \((x, y)\)-plane: one sees that the velocity when the system attains the constraint is tangential to \( x = L + r \), see figure 1.4. As we shall see later, this means that there is no shock in the system when the constraint is attained. This might explain why children can play a long time with yo-yos without breaking them!

**Example 1.5** As another example, consider the system made of a mass \( m \) attached at a fixed point \( O \) by a string of length \( L \). For the sake of simplicity, we also assume only vertical motion. Then the dynamics are:

\[
\begin{align*}
\ddot{x} &= -mg \quad x \leq L
\end{align*}
\]  

(1.33)

Contrarily to the yo-yo, the string exerts no force on the mass as long as \( x < L \). Now when the mass attains its lower position at \( t_0 \) (\( x = L \) and \( \dot{x}(t_0^-) > 0 \)), there is indeed a shock in the system, which corresponds to the reaction between the string and the mass. Thus one sees that although both systems in example 1.4 and in this example are similar one to each other, the kinematic relationships that exist in the yo-yo (the disk rolls without slipping on the string) allow to avoid collisions.
The dynamical systems we deal with can be written as

\[
\begin{align*}
\dot{u} &= G(u) \quad \text{if } t \neq t_k \\
\sigma_u(t_k) &= I_k \left(u(t_k^-)\right) \quad \text{if } t = t_k \\
f(q(t_k)) &= 0 \quad \text{where } u = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}
\end{align*}
\tag{1.34}
\]

The collision times

Important parameters of the dynamics of a mechanical system submitted to unilateral constraints are the collision times, that we denote generically as \( t_k, k \in \mathbb{N} \). Those times are defined such that for some \( \delta_k > 0 \), \( f(q(t)) > 0 \) on \([t_k - \delta_k, t_k)\), and \( f(q(t_k)) = 0 \) with \( \dot{q}(t_k)^T \nabla_q f(q) < 0 \). Suppose that the system has never been in contact with the constraint, so that the first impact time \( t_0 \) is implicitly given by the equation

\[
f_0 \varphi_q(t_0; \tau_0, u_0) = 0 \tag{1.35}
\]

where \( \varphi_q(t; \tau_0, u_0) = q(t) \) is the solution at time \( t \) of the dynamical equation before the impact at \( t_0 \) with vector field \( G(u) \), starting at \( u_0 = (q_0, \dot{q}_0) \) with \( q_0 = q(\tau_0), \dot{q}_0 = \dot{q}(\tau_0) \) and with \( f(q_0) > 0 \). The vector field \( G(u) \) between the impacts can be supposed to be smooth, so that all smoothness properties for solutions of ODE's hold. Assume the constraint is of codimension one (hence \( f(\cdot) \) is smooth in the region of interest). Assume also that the equation in (1.35) possesses at least one solution \( t_0, \tau_0, q_0, \dot{q}_0 \); notice that existence of an impact time depends on \( G(u) \) and on the constraint. In particular for control purposes, one will have to make sure via some suitable feedback control law that the impact times actually exist.

(1.35) provides us with an algebraic relationship between \( t_0, \tau_0, u_0 \) and \( q \), that we can denote as \( h(t_0, \tau_0, u_0) = 0 \). We study this relationship and assume that \( \frac{\partial h}{\partial \tau_0} \) exists for all \( t_0 \), independently of the fact that the solution \( \varphi_q \) will actually not be differentiable at \( t_0 \), but rather possess a left and a right bounded derivatives. We can use the implicit function theorem \([200]\) \([95]\) to deduce that provided \( \frac{\partial h}{\partial \tau_0}(t_0) \neq 0 \), then there is a smooth enough function \( g(\cdot) \) such that \( g(\tau_0, u_0) = t_0 \), and this relation is valid in a neighborhood of \( \tau_0, u_0 \). In other words the set \( h^{-1}(0) = \{(t, \tau, q, \dot{q}) : h(t, \tau, q, \dot{q}) = 0\} \) is a smooth hypersurface of the \((2n + 2)\)-dimensional space of \( t, \tau, q, \dot{q} \) in the neighborhood of \( t_0, \tau_0, q_0, \dot{q}_0 \), defined by the equation \( t = g(\tau, q, \dot{q}) \). Hence the impact time \( t_0 \) clearly depends continuously on the initial data under the stated assumptions\(^{15}\). Notice that it makes no sense to let \( \tau_0 \) tend towards \( t_0 \) this time, contrarily to what we did in section 1.2 to show discontinuity of the solution of a simple MDE with respect to the initial time. Given initial state data.

\(^{15}\)In general, the equation in (1.35) possesses several real solutions, and one has to decide which one is the right one, see e.g. the bouncing ball case in chapter 7, equations (7.19). In the degenerate case, the trajectories in a neighborhood of \( t_0 \) are on the manifold \( \frac{\partial h}{\partial \tau_0}(t_0) = 0, h = 0, f(q) = 0 \) and are tangent to the surface \( f(q) = 0 \) \([48]\).
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and the dynamics, necessarily \( t_0 \geq \tau_0 \), while \( t_0 = \tau_0 \) if the system is initialized on the constraint \( f(q) = 0 \).

**Example 1.6** As an example, consider the dynamics of a ball falling under the influence of gravity on a rigid ground, i.e. \( \ddot{q} = -g \) and \( q \geq 0 \). Then \( \varphi_q(t; \tau_0, q_0, \dot{q}_0) = -\frac{1}{2} (t - \tau_0)^2 + (t - \tau_0)q_0 + q_0 \). Equation (1.35) becomes \( h(t_0, \tau_0, q_0, \dot{q}_0) = -\frac{1}{2} (t_0 - \tau_0)^2 + (t_0 - \tau_0)q_0 + q_0 = 0 \) which possesses two solutions. However only one of them is of interest, such that \( t_0 > \tau_0 \), and given by \( t_0 = \frac{g(\tau_0 + \sqrt{g^2 + 2gq_0})}{g} + \tau_0 \). It is easily checked that \( \frac{\partial h}{\partial t_0}(t_0) \neq 0 \). In this case the function \( g(\tau_0, q_0, \dot{q}_0) \) is defined globally.

Before going on with the other collision times, we need to investigate a little more the properties of solutions.

### 1.3.2 Flows with collisions

**a) Definition**

A natural mathematical interpretation of mechanical systems with unilateral constraints is that of concatenation of flows and mappings: the flows represent the dynamics between the impacts (16), and the mappings are for the relationships between pre and postimpact velocities (i.e. a mapping from the "left" subspace \( f(q) < 0 \) to the "right" subspace \( f(q) \geq 0 \)). We shall study in detail such mappings which are called restitution rules in chapters 4 and 6. Such systems can be called flows with collisions following the terminology in Wojtkowski [586].

We do not have yet presented all the tools that would allow us to correctly describe flows with collisions. However for the sake of completeness of this subsection, let us introduce them in more detail. It is useful, in the body of this presentation, to review some properties of the dynamical systems as in (1.34), in order to make clear the differences between various kinds of MDE's and dynamical equations of mechanical systems with unilateral constraints. We have denoted the domain of the configuration space where the generalized position is constrained as \( \phi \). Let us assume that the boundary of \( \phi \), i.e. \( \partial \phi \), is smooth enough. Firstly, it is more convenient to work with. In particular it allows to properly define restitution rules, i.e. the collision mapping. It implies that the constraint surface can be assumed to be of codimension one, i.e. described by \( f(q) = 0 \) with \( f(q) \in \mathbb{R} \) and \( f(\cdot) \) a smooth enough function. Secondly it will have implications on the uniqueness of solutions.

---

16Recall that given an ODE: \( \dot{x} = f(x) \), its flow is a smooth function of \( t \) and \( x_0 = x(\tau_0) \), denoted as \( \varphi_t(x_0) \), such that \( \frac{\partial \varphi_t(x_0)}{\partial t} = f(\varphi_t(x_0)) \) and with \( \varphi_{\tau_0}(x_0) = x_0 \). In other words, a vector field \( f(x) \) allows the construction of a flow, and the flow is an integral curve of \( f(x) \) (then \( f(x) \) is said to generate the flow \( \varphi_t(x_0) \)). A flow may be local or global, and possesses several properties, like invertibility: \( \varphi_t^{-1}(x_0) = \varphi_{-t}(x_0) \), and the autonomy property: \( \varphi_{t+t}(x_0) = \varphi_t(\varphi_t(x_0)) \). There is a bijective relation between the set of flows and that of generating vector fields. This means that given a priori a flow, there is one and only one vector field that generates it.
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property, that we shall need to prove that the dynamics define a flow \(^{17}\). As we shall see in chapter 2, section 2.2 and in chapter 5 with the sweeping process formulation, it is possible to define a tangent cone to the constraints, that we denote following [381] as \(V(q)\): roughly speaking, this is the half subspace delimited by the hyperplane tangent to the hypersurface \(f(q) = 0\), "inside" \(\Phi\). The symmetric half subspace is denoted as \(-V(q)\). In the following, we assume that the sequence of impacts does exist and that the dynamical system (1.34) possesses solutions on \([\tau_0, +\infty)\) for all \(\tau_0 \geq 0\).

When an impact occurs at \(t_k\), then \(\dot{q}(t_k^-) \in -V(q)\). After the collision, \(\dot{q}(t_k^+) \in V(q)\). Hence the collision mapping \(\mathcal{F}_{q(t_k),k}\) at \(t = t_k\) is defined as

\[
\mathcal{F}_{q(t_k),k} : \partial \Phi \times \{-V(q(t_k))\} \to \partial \Phi \times V(q(t_k))
\]

\[
(q(t_k^-), \dot{q}(t_k^-)) \mapsto (q(t_k^-), \dot{q}(t_k^+)) = u(t_k^+) + I_k \left( u(t_k^-) \right)
\]

(1.36)

We shall assume that the \(\mathcal{F}_{q(t_k),k}\)'s are continuous and autonomous mappings (i.e. the state postimpact value depends only on the state preimpact value). They will not always be invertible for some sort of collisions (named purely inelastic, or soft, or plastic). Note that they are local in nature, because the form of the mapping depends on the system's generalized position at the impact time. In other words, the part of the generalized velocity that jumps at the collision may be modified from one impact to the other, similarly as the tangent cone \(V(q)\) (see for instance (6.37) through (6.39) for an example of configuration-dependent restitution matrices). Indeed, we shall see later that a reasonably sound way to model rebounds between rigid bodies is through restitution rules. Those restitution rules relate post and preimpact velocities, and they can be thought of as composed of two parts: one part is geometrical, i.e. the rule depends on the normal direction to the constraint at the contact point \(q(t_k)\). The second part is energetical, and is given by a coefficient that represents the loss of kinetic energy at the impact. This coefficient may be assumed to be constant, or to vary from one impact to the other as a function of the preimpact normal velocity. The subscripts \(q(t_k)\) and \(k\) for the collision mapping are to recall these two features. Notice that we allow for the moment for different mappings at different collision times. In general however it is accepted that \(\mathcal{F}_{q(t_k),k} = \mathcal{F}_{q(t_k)}\) for all \(k\), i.e. the restitution rule energetical behaviour does not change from one impact to another. We shall make this assumption in the following. We shall also denote

\(^{17}\)In fact, if the constraints are of codimension \(\geq 2\), it is quite possible that all the following developments can be carried out, provided that the set of initial data which yield orbits of the dynamical system striking \(\partial \Phi\) at a singularity, is of zero-measure. Then the flow may be defined in an almost-everywhere sense, as done in [586]. It may however be conjectured [241] that the right way to handle the multiple collisions case, hence being able to define a flow with collisions for a nonsmooth \(\partial \Phi\), is through the extension of the definition of restitution rules, to more general rules, possibly of stochastic nature. This would obviously imply some further mathematical sophistications related to stochastic models. Another path is to choose a particular generalized collision rule, provided that it possesses a coherent mathematical and physical meaning. This is precisely the goal of the sweeping process formulation described in chapter 5, which relies on generalized dissipative shocks.
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\( \mathcal{F}_0(t_0), k \) as \( \mathcal{F}_k \) for simplicity. Let us denote now the flow\(^{18} \) during flight times, defined by \( G(u) \), as \( \varphi_{\tau - \tau_0}(u_0) \). It can be defined either as a mapping from \( \Omega \ni (\tau, u) \) into \( \mathbb{R}^{2n} \) [200], or as a mapping from \( \mathbb{R}^{2n} \) into \( \mathbb{R}^{2n} \) [17]. We adopt the latter definition. \( \varphi_0 \) is the identity map, i.e. \( \varphi_0(u_0) = \varphi(\tau_0; \tau_0, u_0) = u_0 \). Note that \( \tau \) in \( \varphi(\tau; \tau_0, \cdot) \) may denote either the elapsed time from \( \tau_0 \), so that the solution is evaluated at the absolute time \( t = \tau + \tau_0 \), or the absolute time measured from 0. In general (see [95] [200]) one takes \( \tau_0 = 0 \) so that the absolute value of time and the elapsed time are equal. These are matters of convenience. We shall assume in the following that the first argument in \( \varphi(\cdot; \cdot, \cdot) \) is the absolute value of time, so that \( u(t) = \varphi(t; \cdot, \cdot) \) for any initial data.

Just after the first shock, the solution is given at \( t = t_0^+ \) by

\[
\begin{align*}
u(t_0^+) &= \begin{pmatrix} q(t_0) \\ \varrho(t_0^+) \end{pmatrix} = \mathcal{F}_0 \begin{pmatrix} \varphi_q(t_0; \tau_0, u_0) \\ \varphi(\varrho_0(t_0; \tau_0, u_0)) \end{pmatrix} = \mathcal{F}_0 \varphi_{\tau - \tau_0}(u_0) \quad (1.37)
\end{align*}
\]

Recall that \( f \circ \varphi_q(t_0; \tau_0, u_0) = 0 \). Notice that the expression \( \mathcal{F}_k \circ \varphi_{\tau - \tau_0}(u) \) is meaningful as soon as \( t \) is a collision time. Actually, the solution \( u(t) \) is obtained by the time-concatenation of the successive values of \( \varphi(\cdot; \cdot, \cdot) \) considered as a function of time from \( \tau_0 \) to \( t_0^+, t_0^+ \) to \( t_0^+ \), \( t_0^+ \) to \( t_1^+ \), ... In terms of flow we have the composition \( \varphi(t; \tau, \varphi(\tau_1; \tau_0, u_0)) = \varphi_{t - \tau}(\varphi_{\tau_1 - \tau_0}(u_0)) = \varphi_{t - \tau} \circ \varphi_{\tau_1 - \tau_0}(u_0) \).

Now during the flight time after the first shock, i.e. on an interval \( (t_0, t_0 + \delta) \) for some \( \delta > 0 \), the solution continues to evolve with new initial data \( q(t_0), \varrho(t_0^+) \) according to the vector field \( G(u) \). The solution on this interval is given by \( \varphi(t; \tau_0, u_0) = \varphi(t; t_0^+, u(t_0^+)) = \varphi(t; t_0^+, \mathcal{F}_0 \circ \varphi(t_0^-; \tau_0, u_0)) \). Hence with the above notation \( \varphi_{t_0^-}(u_0) = \varphi_{t_0^- - t_0^+}(u(t_0^+)) = \varphi_{t_0^- - t_0^+}(\mathcal{F}_0 \circ \varphi(t_0^-; \tau_0, u_0)) = \varphi_{t_0^-} \circ \mathcal{F}_0 \circ \varphi_{t_0^- - \tau_0}(u_0) \).

The superscript in \( \varphi_{t_0^-} \) is to indicate for the moment that this denotes the solution after one shock has occurred. Proceeding similarly, after the second shock at \( t_1 \), we can write \( \varphi_{t_1^-}(u_0) = \varphi_{t_1^- - t_0^+} \circ \mathcal{F}_1 \circ \varphi_{t_1^- - t_0^+} \circ \mathcal{F}_0 \circ \varphi_{t_0^- - \tau_0}(u_0) = \varphi_{t_1^-} \circ \mathcal{F}_1 \circ \varphi_{t_1^- - \tau_0}(u_0) \). We denote now the solution on \( (t_k, t_{k+1}) \) starting at \( (\tau_0, u_0) \) as \( \varphi_{t_0^-} \). Note that \( \varphi_{t_0^-}(u_0) = \varphi(v_0(u_0) = \varphi(\tau_0; \tau_0, u_0) = u_0 \) is the identity mapping. The candidate flow with collisions \( \varphi_{t_0^-} \) is thus defined on \( (t_k, t_{k+1}) \) with \( \tau_0 = 0 \) as

\[
\varphi_t(u_0) = \varphi_{t - t_k^-} \circ \mathcal{F}_k \circ \varphi_{t_k^- - t_k^- - t_k^-} \circ \mathcal{F}_{k-1} \circ \varphi_{t_k^- - t_k^-} \circ \mathcal{F}_{k-2} \circ \varphi_{t_k^- - t_k^-} \circ \mathcal{F}_0 \circ \varphi_{t_0^-}(u_0) \]

\[
= \varphi_{t - t_k^-} \circ \mathcal{F}_k \circ \varphi_{t_k^- - t_k^-}^{-1}(u_0) \quad (1.38)
\]

\(^{18}\)It is justified to speak of the flow between impacts since the dynamics are smooth during those period. Whether the total dynamics (smooth plus jumps) defines a flow remains at this step an open question.
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that is

\[ \varphi_\varepsilon^c : \mathbb{R}^{2\varepsilon} \rightarrow \partial \Phi \times \{-V(q(t_0))\} \rightarrow \partial \Phi \times V(q(t_0)) \rightarrow \partial \Phi \times \{-V(q(t_1))\} \rightarrow \ldots \]

\[ \rightarrow \partial \Phi \times \{-V(q(t_k))\} \rightarrow \partial \Phi \times V(q(t_k)) \rightarrow \mathbb{R}^{2\varepsilon} \]

\[ u_0 \mapsto \varphi(t_0^-; 0, u_0) \mapsto \varphi(t_0^+; 0, u_0) \mapsto \varphi(t_1^-; 0, u_0) \mapsto \ldots \mapsto \varphi(t_k^-; 0, u_0) \]

\[ \mapsto \varphi(t_k^+; 0, u_0) \mapsto \varphi(t; 0, u_0) \]

(1.39)

b) The autonomy property

Recall that in order to prove that the solution \( \varphi(t; \tau_0, u_0) \) actually defines a flow, we must prove that the autonomy property \([95]\) is verified, i.e. the solution \(( \in \mathbb{R}^{2\varepsilon} \)) verifies

\[ \varphi(t_2 + t_1; 0, u_0) = \varphi(t_2; 0, \varphi(t_1; 0, u_0)) \]

for all \( t_2 \) and \( t_1 \): The solution at time \( t_2 + t_1 \), starting at \( t = 0 \), with initial condition \( u_0 \), and the solution at time \( t_2 \) with initial condition \( \varphi(t_1; 0, u_0) \), coincide. Equivalently the autonomy property can be stated as

\[ \varphi(t_1; 0, u_0) = \varphi(t_1 - t_0; 0, u_0). \]

The solution at time \( t_1 \) starting at time \( t_0 \) with initial data \( u_0 \), is equal to the solution at time \( t_1 - t_0 \), starting at \( t = 0 \) with the same initial state data, for any \( t_0, t_1 \). Uniqueness of solutions is the key property. In other words, autonomy means that the absolute value of the initial time is not an important notion. It is rather the elapsed time which has to be considered to compute the solution from a set of initial data \([17]\). For the sake of simplification of the notations, in the following we shall denote \( \varphi(t; 0, u_0) \) as \( \varphi(t, u_0) \).

Our goal is therefore to show that if \( \varphi(t + \tau, u_0) \) and \( \varphi(t, \varphi(\tau, u_0)) \) are two solutions of the system, they are equal for all \( t \geq 0 \). The first impact time \( t_0 \) is given by \( f \circ \varphi_\varepsilon(0, u_0) = 0 \). Note that \( f \circ \varphi_\varepsilon(t + \tau, u_0) = 0 \) for \( t + \tau = t_0 \), i.e. \( t = t_0 = t_0 - \tau \).

If \( \tau \leq t_0 \), then \( \varphi(t, \varphi(\tau, u_0)) \) jumps at \( t_0 \) also, and \( \varphi(t_0^-, \varphi(\tau, u_0)) = \varphi(t_0^- + \tau, u_0) \) because the equality is true before any jump occurs. Then we deduce that

\[ f_0 \circ \varphi(t_0^-, \varphi(\tau, u_0)) = f_0 \circ \varphi(t_0^- + \tau, u_0), \]

i.e. \( \varphi(t_0^+, \varphi(\tau, u_0)) = \varphi(t_0^+ + \tau, u_0) = \varphi(t_0^+ - \tau, u_0) = \varphi(t_0^+ - \tau, u_0) = \varphi(t_0^+ - \tau, u_0) \). We deduce that \( \varphi(\tau, u_0) = \varphi(\tau, u_0) \) for all \( t \geq 0 \) (until an eventual second impact occurs).

If \( \tau > t_0 \), the first impact time does not change but \( \varphi(\tau, u_0) \) has already jumped. Hence we can attack the proof assuming that the autonomy property is true from \( t = 0 \) until the second impact at \( t_1 \), \( t_1 > t_0 \) and with \( f \circ \varphi_\varepsilon(t_1, u_0) = 0 \). Assume that \( \tau < t_1 \). Then we can redo the same reasoning as above, replacing \( t_0 \) by \( t_1 \) and \( f_0 \) by \( f_1 \).

The reasoning can be extended for any \( \tau > 0 \). The fact that both functions \( \varphi(t + \tau, u_0) \) and \( \varphi(t, \varphi(\tau, u_0)) \) take the same values for all \( t \) and \( \tau \) relies at each

\[ ^{19} \text{Clearly this property is not true in general for nonautonomous systems, since the initial vector field, i.e. the slope of the curve (the orbit) in the 2-dimensional case, changes if the initial time changes. Therefore even if the initial state remains unchanged, there is no reason that after a certain amount of time, both solutions coincide} \]
step that there is a unique solution to the dynamical problem considered, for given initial data. As we shall see in chapter 2, uniqueness for problems with unilateral constraints fail in general if no restrictions are placed on the vector field $G(u)$, on the hypersurface $f(q)$ and on the collision mapping energetic behaviour (related to the invertibility of $F_k$) [476] [94] [427] [428]. We have assumed from the beginning that the vector field $G(u)$ is autonomous. This means that the possible external bounded actions on the system are constant on $[\tau_0, +\infty)$. If there are no losses of kinetic energy at impacts the results in [427] allow to conclude that $\varphi_t^k$ is a flow. It is noteworthy that the time-independence of both the vector field and the collision mapping is not sufficient to guarantee this result in general. In conclusions, the only mechanical systems with unilateral constraints which can be said to define a flow are for the moment restricted to those that fit within the framework developed by Percivale in [427] [428], see theorem 2.7. This in particular precludes the existence of finite accumulation points for the sequence \{$t_k$\}. In other words the solutions are in $RCLBV$ but may simply be considered as piecewise continuous time functions.

Contrarily to the case of autonomous ODE's, here $\varphi_t^k$ cannot be continuous in $t$. It can be expected to be $RCLBV$ (or piecewise continuous) in $t$. Furthermore in order for $\varphi_t^k$ to be a global flow, it must exist the inverse function $(\varphi_t^k)^{-1} = \varphi_t^{-k}$ for all $t \geq \tau_0$. Invertibility of the $F_k$'s is then necessary, since one has

$$
(\varphi_t^k)^{-1} = \varphi_t^{-k} = \varphi_{\tau_0-\tau_k}^{-1} \circ F_0^{-1} \circ \varphi_{\tau_0-\tau_k-1}^{-1} \circ \ldots \circ \varphi_{\tau_0}^{-1} \circ F_k^{-1} \circ \varphi_{\tau_0-t_k}^{-1} t_k \tag{1.40}
$$

This places the dynamics of so-called soft or inelastic shocks (in a sense the shock produces a maximal loss of kinetic energy) well apart from those of hard or elastic shocks (the loss of kinetic energy is zero)\footnote{It is interesting to note that most of the mathematical literature on the subject treats these two extreme cases and leaves the intermediate cases apart, as we shall describe in chapters 2 and 5.}. For instance, in the case of the bouncing ball, we get $F(q(t_k), \dot{q}(t_k)) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q(t_k) \\ \dot{q}(t_k) \end{pmatrix}$ for the soft case, and

$$
F(q(t_k), \dot{q}(t_k)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} q(t_k) \\ \dot{q}(t_k) \end{pmatrix}
$$

for the elastic case.

c) Continuity of solutions in the initial data

We have seen in section 1.2 that the solutions of MDE's as defined in (1.13) are not continuous but of bounded variation in the initial time $\tau_0$. They are however continuous in initial state data, and this is easily proved since the jump times are exogeneous, hence equal for any trajectory. What about continuity with respect to $\tau_0, u_0$ of the solutions $\varphi(t; \tau_0, u_0)$ of mechanical systems with unilateral constraints? Let us recall that we assume a codimension one constraint, and that we suppose that solutions exist\footnote{Existence is evidently the basic property that should be proved before the rest, and we shall come back on existential results in next chapters. We take some freedom here with the mathemat-}. We shall see later on that when there are several constraints (or a constraint of codimension \geq 2), things are more involved.
1.3. SYSTEMS SUBJECT TO UNILATERAL CONSTRAINTS

On \([\tau_0, t_0^+\) continuity holds. Now at \(t_0^+\), \(\varphi(t_0^+; \tau_0, u_0) = \mathcal{F}_0 \varphi(t_0^-; \tau_0, u_0)\). From the continuity of \(t_0\) in \(\tau_0\) and \(u_0\), it follows that the last term can be written as \(\mathcal{F}_0 \varphi(t_0^-; \tau_0, u_0)\) for some continuous function \(m(\cdot, \cdot)\). Now since \(\mathcal{F}\) is continuous, it follows that \(\varphi(t_0^+; \tau_0, u_0)\) is continuous in \(\tau_0\) and \(u_0\). Therefore on \((t_0, t_1)\), the solution \(\varphi(t; t_0^+, u(t_0^+))\) is also continuous in the initial data. Now the collision time \(t_1\) is given by \(f_1(t_1; t_0, u(t_0)) = 0\). Using similar arguments as in subsection 1.3.1 we deduce that \(t_1\) depends continuously on \(t_0\) and on \(u(t_0)\), hence on \(\tau_0, u_0\). Thus on \((t_1, t_2)\), \(\varphi(t; \tau_0, u_0) = \varphi(t; t_1^+, \mathcal{F}_0 \varphi(t_1^-; \tau_0, u_0))\). Since \(t_1\) depends continuously on \(\tau_0, u_0\), the term \(\mathcal{F}_0 \varphi(t_1^-; \tau_0, u_0)\) is a continuous function of \(\tau_0, u_0\), and so is \(\varphi(t; t_1^+, u(t_1^+))\). Reiterating this reasoning, we deduce that the solution depends continuously on \(\tau_0, u_0\) on intervals \((t_k, t_{k+1})\).

Let us now examine what happens at the jump times. Let us further suppose first that \(t_{k+1} > t_k + \delta\) for some \(\delta > 0\), i.e. velocities are piecewise time-continuous. In order for the solution to be a continuous function of \(u\) at \(u_0\), it must be verified that for all \(t\) and for all \(\tau_0\), for any \(\varepsilon > 0\), there exists a \(\delta > 0\) such that for all \(u_1\) with \(||u_1 - u_0|| < \delta\), then \(||\varphi(t; \tau_0, u_1) - \varphi(t; \tau_0, u_0)|| < \varepsilon\). Let us consider \(u_0\) such that \(f(q_0) = 0\) and \(q_0 \in -V(q_0)\). In other words the system is initialized on the constraint and with a velocity pointing outwards \(\Phi\): a shock occurs at \(t_0 = \tau_0\) and the solution \(\varphi(t; \tau_0, u_0)\) jumps at \(\tau_0\). Now consider \(u_1\) with \(f(q_1) = \mu > 0\) and \(q_1 \in -V(q_0)\). The solution may or may not jump, but anyway if it does, then it jumps at a time \(t_0 > t_0\) since the system has to attain the constraint. Hence \(\varphi(t; \tau_0, u_1)\) is continuous (in \(t\)) at \(t = t_0\). The quantity \(||\varphi(t_0^+; \tau_0, u_1) - \varphi(t_0^+; \tau_0, u_0)||\) thus cannot be made arbitrarily small even for an arbitrarily small \(\mu > 0\). We conclude that for such a \(u_0\), there exist \(\varepsilon > 0\) and \(t \geq \tau_0\) such that for any \(\tau_0\) and for any \(\delta > 0\), there exists \(u_1\) with \(||u_1 - u_0|| < \delta\) and \(||\varphi(t; \tau_0, u_1) - \varphi(t; \tau_0, u_0)|| > \varepsilon\). Another way of seeing this fact is to consider sequences \(\{u_n\}\) that converge towards \(u_0\). Then \(\varphi(t; \tau_0, u)\) is continuous at \(u_0\) if and only if, for any such sequence \(\{u_n\}\), \(\varphi(t; \tau_0, u_n)\) converges towards \(\varphi(t; \tau_0, u_0)\), for all \(t \geq \tau_0\) and for all \(\tau_0\). If this is true, then for all \(\varepsilon > 0\), there exists \(N > 0\), \(N \in \mathbb{N}\), such that \(n > N\) implies \(||\varphi(t; \tau_0, u_n) - \varphi(t; \tau_0, u_0)|| < \varepsilon\), for all \(t, \tau_0\). In particular this must hold at \(t = t_0^+\) as defined above (the first time of jump for \(\varphi(t; \tau_0, u_0)\)). Consider for instance \(q_n\) such that \(f(q_n) = f(q_0) + \frac{1}{n} = \frac{1}{n}\). Then clearly there exists \(\varepsilon > 0\) such that for any \(N > 0\), there exists \(n > N\) such that \(||\varphi(t_0^+; \tau_0, u_n) - \varphi(t_0^+; \tau_0, u_0)|| > \varepsilon\). Notice that if we estimate the solutions at \(t \geq t_0 + \alpha, \alpha > 0\), then it is always possible to find \(u_1\) so close to \(u_0\) that \(\varphi(t; \tau_0, u_1)\) has jumped before such \(t\). This is possible since the jump times depend continuously on the initial data. In other words, \(||\varphi(t_0^+ + \alpha; \tau_0, u_0) - \varphi(t_0^+ + \alpha; \tau_0, u_0)||\) can be made arbitrarily small by taking \(n\) sufficiently large but finite. Indeed the first impact time \(t_0n\) associated to the solution initialized at \(\tau_0\) (i.e. \(u_n\)) can be made arbitrarily close to \(t_0\) by increasing \(n\). We retrieve the fact that if \(t \neq t_0\), then the solution is continuous in \(u_0\). This motivates us to define the closeness of two solutions as follows, where \(t_k\) denotes the
discontinuities of \( \varphi(t; \tau_0, u_0) \):

\[
\forall \varepsilon > 0, \forall \alpha > 0, \exists \eta > 0 \text{ such that } |t - t_k| > \alpha \text{ and } ||u_0 - u_1|| \leq \eta
\]

\[
\Rightarrow ||\varphi(t; \tau_0, u_0) - \varphi(t; \tau_0, u_1)|| \leq \varepsilon
\]  

(1.41)

In other words, by letting \( u_1 \) tend towards \( u_0 \), both solutions become arbitrarily close one to each other, except in a neighborhood of their discontinuities. The size of this neighborhood (i.e. \( \alpha \)) can be decreased arbitrarily by decreasing \( \eta \).

In conclusion the solutions are continuous in the initial data, in the sense that

\[
\lim_{u_1 \to u_0} \varphi(t; \tau_0, u_1) = \varphi(t; \tau_0, u_0).
\]

But the fact that basically, the dynamical system consists of flows and diffeomorphisms implies some modifications of the "continuity" definition, taken from the continuous-time point of view only.

**Remark 1.9** We shall retrieve in the definition of stability of trajectories (see definition 7.1) that two solutions cannot be arbitrarily close one to each other in the neighborhood of the discontinuity times. Hence the classical Lyapunov stability definition has to be modified. It is known (see [554]) that continuity with respect to initial conditions and stability are closely related, for solutions of ODE's. Hence it is not surprising that both notions are related also for MDE's representing systems with unilateral constraints.

**Remark 1.10** The results on continuous dependence on the initial data are similar to those in [27] theorem 3.9 and theorem 3.10 which concern systems as in (1.19) with either state dependent or fixed discontinuities times.

What happens now if the sequence \( \{t_k\} \) is infinite and with a finite accumulation point \( t_\infty \)? Although the above reasoning applies well for \( t < t_\infty \), it is not clear how we should study the behaviour of the solution at \( t_\infty \). Indeed the criterion in (1.41) does not apply well in the limit as \( k \to +\infty \), because it is no longer possible to define neighborhoods of the impact times (\( \alpha \) is strictly positive in (1.41)). One point of view is to do a sort of time-scaling as follows. Since the sequence is infinite and with \( t_\infty < +\infty \), the flight-times are bounded and the system's state remains bounded between each impact. Hence the total dynamics define (explicitly or implicitly) an operator (or a mapping) \( P : (u(t^+_k), t_k) \mapsto (u(t^+_k+1), t_k+1) \)\(^{22}\). Let us denote \( (u(t^+_k), t_k) \) as \( x_k \). Then \( x_k = P^k(x_0) \). We can therefore consider the finite collisions process as an infinite discrete-time system in the \( k \)-time scale. From the above developments it is clear that \( x_k \triangleq x_k(x_0) \) is continuous in \( x_0 \) for all finite \( k \).

If it can be proved for instance that the sequence of continuous functions \( \{x_k(\cdot)\} \) converges uniformly towards a limit \( x(\cdot) \), then \( x(\cdot) \) is continuous. Clearly it is not sufficient that continuity holds for any finite \( k \) to imply that it holds also at the limit, which might be discontinuous.

\(^{22}\) which could as well be defined with preimpact values.
d) Controllability properties

How does the addition of a unilateral constraint and an associated impact map $\mathcal{F}_k$ influence a property like controllability of the system? It is clear that any state $u$ with $f(q) < 0$ (i.e. outside the domain $\Phi$) cannot be attained by the system. Now if the (possibly nonlinear) continuous dynamics are controllable (see [399] definition 3.2), the subspace $\text{Int}(\Phi) \times \mathbb{R}$ is controllable also. The point here is that the collisions are uncontrolled, since the mapping $\mathcal{F}_k$ does not contain any control action (if it did, this control would be necessarily impulsive, but this poses some problems as explained just below). Consider now a state $u_0$ with $q_0 \in \partial\Phi$ and $q_0$ pointing outwards $\Phi$. Does there exist some states that cannot be reached from $u_0$? This could be the case if the mapping $\mathcal{F}_0$ is not onto (the image of $\mathcal{F}_0$ is not the whole of $\partial\Phi \times V(q_0)$). If those states cannot be reached from any state in $\text{Int}(\Phi)$, controllability in $\Phi \times \mathbb{R}$ is lost. For instance it is clear that a state $u_1$ with $q_1 \in \partial\Phi$ and $q_1$ pointing inwards $\Phi$ cannot be reached from $\text{Int}(\Phi)$ in the case of a plastic collision, since by definition such a collision process is such that the postimpact velocity is tangential to $\partial\Phi$, see the one-degree-of-freedom example given just below. If the collision mapping $\mathcal{F}_0$ is diffeomorphic (at least possesses an inverse mapping), it is always possible to attain a state like $u_1$ by first reaching the suitable preimpact state $\mathcal{F}_0^{-1}(u_1)$. But if it is not, there may be some states like $u_1$ that cannot be attained at $t_k$. Consider for instance the simple one-degree-of-freedom controlled bouncing ball example with $q > 0$, and purely dissipative shocks, i.e. $\dot{q}(t_k^+) = 0$. Then a state $q = 0$, $\dot{q} > 0$ cannot be attained. One might be tempted to think that an impulsive input $p\delta_t$ could be used to allow to attain such state. But notice that the fact that the collision is purely inelastic is a physical law. As we shall see in the next chapters, this can be stated through a so-called restitution rule: in the one degree-of-freedom case, one has $\dot{q}(t_k^+) = -e\dot{q}(t_k^-)$, where $e = 0$ for inelastic shocks. Assume that the impulsive input $p\delta_t$ is applied so that the postimpact velocity has strictly positive magnitude and points inwards $\Phi$. Then the dynamics yield $m(\dot{q}(t^+) - \dot{q}(t^-)) = p$. Clearly if $\dot{t} = t_k$, there is a contradiction since $\dot{q}(t_k^+) = 0$ cannot be zero and nonzero simultaneously! Hence one deduces that the impulse input is permitted at all $\dot{t} \geq t_k + \varepsilon$, with $\varepsilon > 0$. But strictly speaking, the restitution rule implies that $\dot{q}(t_k^+) = 0$, and this cannot be modified via the input. It is a general feature that one must take care of not defining impulsive inputs at impact times, otherwise encountering a non-wellposed dynamical system (23). However distributional inputs are permitted on $\partial\Phi$ for $t > t_k$.

We can state the following sufficient condition:

**Claim 1.2** Assume that the system is controllable in $\text{Int}(\Phi) \times \mathbb{R}$, and that the collision mapping $\mathcal{F}_q$ is invertible for all $q \in \partial\Phi$. Then the total system (i.e. the flow with collisions) is controllable.  

---

23Recall [399] definition 3.2, that controllability is tested with admissible inputs, i.e. inputs such that the closed-loop system is wellposed. This is the least requirement on inputs of controlled systems. Clearly here the set of admissible inputs $\mathcal{U}$ excludes Dirac measures at times $t_k$. 

---
Controllability properties of systems with unilateral constraints need to be investigated more. Indeed consider for instance juggling and catching tasks (see chapter 8 for details). Both systems are such that the free-motion dynamics are not completely controllable (only one part is, i.e., the robot's state). But the uncontrollable part (the object) may be submitted to gravity (juggling) or evolve without any external action (catching). In the first case one intuitively conceives that it is possible to strike the object so that it tracks a certain path (periodic trajectories for instance). In the second case the possible motions of the ball are much more restricted: one may only plan to extract some kinetic energy from the object to stabilize it on the robot. Consequently one clearly needs to refine the properties of the uncontrollable part to be able to characterize the "controllability" of the overall system. It is also possible that an interesting concept is that of controllability of only one part of the system (here the object), through collisions with the constraint. A path to follow could be to first characterize the controllability property of the uncontrollable part through general impulsive inputs; then to restrict those inputs to the percussions occurring in the system (and thus take into account not only the controllability of the remaining part, but also the unilateral constraints dynamics: indeed the trajectory between two prespecified impacts plays an important role contrarily to free-motion systems. Imagine that an input takes the robot's state from one value $u_0$ to $u(t_k^r)$, where $u(t_k^r)$ is computed to yield a desired object's state. One must guarantee that no collision occurs on $[0, t_k)$, otherwise the whole scheme fails).

Remark 1.11 The nonsmooth coordinate change described in subsection 1.4.2 (that we call the Zhuravlev-Ivanov coordinate transformation) may also be used in this context to check controllability properties. It possesses the advantage of eliminating the Dirac measures from the dynamical equations. The obtained system has a state discontinuous right-hand-side, hence continuous trajectories in the sense of Filippov [154], and is closer to "classical" systems than MDE's. It is worth noting that as we shall see, the nonsmooth transformation is not defined for plastic impacts. This might be related to what we noticed above on the particular feature of such systems.

1.3.3 Constrained systems and descriptor variable systems

We have started from (1.1), i.e., from the interaction force model to show the impulsive behaviour of two rigid bodies that collide, and the consequence on the obtained differential equations. It is worth noting that similar conclusions (on the necessary impulsive behaviour) for example 1.1 could have been obtained using the point of view of descriptor variable systems [110] adopted for example in [4] [215] [325] [357] [492]. These systems are of the form $E\dot{x} = Ax + Bu$ where $E$ is singular. The solutions of such differential equations possess a jump in the initial condition, i.e., in general $x(0^-)$ and $x(0^+)$ are not equal [111]. It is possible to decompose such systems as [110] [111]

$$\dot{x}_s = L_s x_s + B_s u$$

(1.42)
and

\[ L_f \ddot{x}_f = x_f + B_f u \quad (1.43) \]

where \( L_f \) is a nilpotent matrix with index of nilpotency \( p \), i.e. \( L_f^p = 0 \). Then the solution can be written as \( x = x_s + x_f \), where

\[ x_s = \exp(tL_s)x_0 + \exp(tL_s) * B_s u \quad (1.44) \]

where * is the convolution product, and

\[ x_f = -\sum_{i=1}^{p-1} \delta_0^{(i-1)} L_f^i x_{0f} - \sum_{i=0}^{p-1} L_f^i B_f u(i) \quad (1.45) \]

One sees that in general, \( x_f \) possesses a jump at \( t = 0 \). Initial jumps in the solutions exist because of inconsistent initial conditions [317]. In the case of mechanical systems subject to holonomic constraints, the initial conditions can be chosen in accordance with the constraint so that no impulsive behaviour occurs. The point of view of singular systems is used in [357] [362] who propose a quite interesting and detailed application of the theory in [110] [111] to robotic systems with a linearized model of a \( n \)-degree-of-freedom manipulator with kinematic constraints (the available theory for control of descriptor variable systems is apparently restricted to linear time-invariant systems). By linearizing the dynamics of a \( n \)-degree-of-freedom rigid manipulator around a nominal trajectory \( q_0, \dot{q}_0 \) and \( \lambda_0 \) (\( \lambda_0 \) being the nominal value of the Lagrange multipliers associated with a set of holonomic constraints \( f(q) = 0 \)), Mills and Goldenberg [362] transform the nonlinear constrained system into

\[
\begin{bmatrix}
I & 0 & 0 \\
0 & M(q_0) & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta \dot{q} \\
\Delta \ddot{q} \\
\Delta \dot{\lambda}
\end{bmatrix}
=
\begin{bmatrix}
0 & I & 0 \\
\frac{\partial f}{\partial q}(g(q_0) - \frac{\partial f}{\partial q}(q_0)\lambda_0) & 0 & \frac{\partial f}{\partial q}(q_0) \\
\frac{\partial f^T}{\partial q}(q_0) & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta q \\
\Delta \dot{q} \\
\Delta \dot{\lambda}
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0 \\
g(q) - g(q_0)
\end{bmatrix}
\Delta U
\]

where \( \Delta(\cdot) \triangleq (\cdot) - (\cdot)_0 \). \( \Delta U \) is obtained as the difference between the Taylor expansion of the control torque around the nominal trajectory and a nominal controller \( U_0 = g(q_0) - \frac{\partial f^T}{\partial q}(q_0)\lambda_0 \). Hence the system can be decomposed as in (1.42) (1.43).

It is shown in [357] on a particular numerical example that no control bounded torque \( U \) exists such that the impulses can be eliminated for arbitrary initial conditions (the only solution being to keep the contact between the two bodies all the
CHAPTER 1. DISTRIBUTIONAL MODEL OF IMPACTS

time). Note that this is a reassuring conclusion: the opposite result would have meant that even if the system is initialized on the constraint and with a velocity pointing outwards the domain \( \Phi \) of possible motions, then one could find a feedback control that would eliminate the velocity jump. This is quite impossible as we have seen that a bounded controller has no effect on the system at the impact times. As we shall see when we deal with control, one can only influence the motion between collisions. The work in [4] lemma 3.2 permits to state the result in [357] formally. Let us notice that the descriptor system approach might also be used to deduce the fact that contact percussions are Dirac measures and thus retrieve a similar conclusion as ours, although our conclusion was physically motivated. Proving this for any output or state feedback could be a subject of future work, as well as designing output feedback control laws in order to modify the impulsive forces, for instance minimize them. Other approaches have considered this problem but with quite different point of views, as we shall see in a next section on control of percussive systems. Still the link with Bressan's *hyper-impulsive* systems could be investigated. It is also noteworthy that the study in [110] does not answer to the issue raised in example 1.1 below claim 1: Cobb [110] chooses the singular perturbations approach to approximate descriptor variable systems, shows that reasonable approximating sequences \(^{24}\) exist and yield a unique limit. The dynamics are understood in a distributional sense, i.e. in \( \mathcal{D}^* \) (see appendix A). When the results are applied to constrained mechanical systems, the only thing that one may think is that among the approximating sequences there may be compliant environments of any kind, but nothing more. The study of numerical integration of descriptor variable systems (also known as DAE, i.e. Differential-Algebraic Equations) has been investigated in [167]. Finally let us note that it is not clear how singular systems might be used to study a general nonsmooth mechanical problem (where there may exist particular behaviours such as finite accumulation of discontinuities).

1.4 Changes of coordinates in MDE

Now that we have set the relationship between external impacts and unilateral constraints, let us investigate some tools to eliminate the singular distributions from MDE's. The goal of these analytical tools is to investigate more easily the behaviour of trajectories in some vibro-impact systems. In fact they allow to write the dynamical equations of impacting systems without the use of singular distributions. This fact may prove to be useful in certain cases.

1.4.1 From measure to Carathéodory systems

Note that we could have proceeded in a different way to solve the dynamics of the system in example 1.1, equation (1.2). Let us write the state space equations for

\(^{24}\)i.e. sequences with solutions that do not pointwisely converge to \(+\infty\) for all \( t \in (0, +\infty) \) [110] section 3.
1.4. CHANGES OF COORDINATES IN MDE

this system:

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
x_2 \\
0
\end{pmatrix} + \begin{pmatrix}
0 \\
\frac{\partial h}{\partial x_1}
\end{pmatrix} h
\]

(1.47)

where \( h(t) \equiv 0 \) for \( 0 \leq t < t_k \), \( h(t) \equiv 1 \) for \( t_k \leq t \), \( x_1 = x \), \( x_2 = \dot{x} \). Following the ideas in [154] on change of variables in differential equations with distributions in coefficients, let us consider now \( y = x_2 - \frac{\partial h}{\partial x_1} h \); then in the \((x, y)\) coordinates (1.47) becomes:

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{y}
\end{pmatrix} = \begin{pmatrix}
y \\
0
\end{pmatrix} + \begin{pmatrix}
\frac{\partial h}{\partial x_1} \\
0
\end{pmatrix} h
\]

(1.48)

from which it follows that \( y \equiv y_0 \), \( \dot{x}_1 = y_0 + \frac{\partial h}{\partial x_1} h \). Thus \( x_2 = \dot{x} = \frac{\partial h}{\partial x_1} h + y_0 \) and we retrieve the preceding results, i.e. the velocity \( \dot{x} \) is discontinuous at \( t_k \). Note that \( \dot{x}(t_0) = \frac{\partial h}{\partial x_1} h + y_0 \) so that if \( t_0 < 0 \), \( \dot{x}(t_0) = y_0 \). Now if \( t_0 = 0 \), we set \( \dot{x}(0^-) = y_0 \) so that \( \dot{x}(0^+) = \frac{\partial h}{\partial x_1} + \dot{x}(0^-) \).

Still following [154] we can proceed as above to draw conclusions about existence and uniqueness of solutions. Notice that we can write (1.10) as follows:

\[
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= a(x_1, x_2, t) + b(x_1) h
\end{aligned}
\]

(1.49)

where \( a(\cdot, \cdot, \cdot) \) and \( b(\cdot) \) have obvious definitions, \( h \) is as in (1.47). Now take \( y_2 = x_2 - b(x_1) h \), then (1.49) becomes:

\[
\begin{aligned}
\dot{x}_1 &= y_2 + b(x_1) h \\
\dot{y}_2 &= a(x_1, y_2, t) - \frac{\partial h}{\partial x_1}(x_1)(y_2 + b(x_1) h) h
\end{aligned}
\]

(1.50)

The equation in (1.50) satisfies the Carathéodory conditions on existence and uniqueness of solutions. Recall that considering (1.50) it is possible to assign an initial arbitrary value to \( y_2 \) at discontinuities of \( h(t) \), but since \( x_2(t) = y_2(t) + b(x_1(t)) h(t) \), this is not possible for \( x_2 \) as \( h(t) \) is not defined at those times. Only left and right limits can be assigned to \( x_2(t) \). Initial data outside the discontinuities of \( h(t) \) form a zero-measure set as long as \( h \) is of bounded variation. If we write it shortly as \( \dot{z} = g(z, t) \) then in a domain \( S \) of the \((t, x)\)-space

- The function \( g(z, t) \) is defined and continuous in \( z \) for almost all \( t \).
- The function \( g(z, t) \) is measurable in \( t \) for each \( z \).
- \( |g(z, t)| \leq m(t) \) for some measurable function \( m(t) \).

Thus there exists a maximal solution \( z(t) \) to the system in (1.50) and this solution is a continuous time function [154] chapter 1. Therefore assuming the control
CHAPTER 1. DISTRIBUTIONAL MODEL OF IMPACTS

input u has been suitably designed, there is no finite escape time in the system, x₁ and y₂ are continuous, and x₂ jumps at the instant of the percussion.

Thus clearly such mechanical systems excited by impulses belong to the class of systems with singular distributions in coefficients that can be reduced to a Carathéodory (ordinary) system by a change in the unknown function (see [154] for other examples of such manipulations). Let us see a possible application of such manipulations. Assume that the control input u has been chosen along a certain "high-level" switching strategy, so that at certain times tₘ it switches from one value to another one, and the switching mechanism lasts a period Δ (Such complex control strategies have been chosen in the recent robot control literature to cope with control of manipulators during contact-non contact tasks [360]). Then it is reasonable to consider that on Δ [tₛ, tₛ + δ], u ∈ [uₘ, u_M], where uₘ and u_M may be chosen as 

\[ uₘ = \min_{t \in \Delta} u(t), \quad u_M = \max_{t \in \Delta} u(t). \]

Therefore (1.49) is basically a differential inclusion on Δ, \( \dot{X} \in G(X,t) \), together with some jump condition on \( \dot{x} \). Note that here we prefer to write the system as a separated smooth inclusion together with jump conditions than something like \( \dot{X} \in G(X,t, δₜₖ) \). Indeed it is not clear which is the meaning of such right-hand-side. Consequently (1.50) may be written as \( \dot{z} \in F(z,t) \), where \( F(z,t) \) is a subset of the state space. For instance in the 2 degree-of-freedom case, it is a vertical segment with abscissa \( y₂ + b(x₁)h \) (note that its length is given by \( |a(x₁, uₘ) - a(x₁, u_M)| \) in case \( u \mapsto a(x₁, u) \) is a monotonic function). If there are impacts on Δ, then \( F(\cdot, \cdot) \) is discontinuous in \( t \), and continuous in \( z \). Now assume it can be proved that \( h(t) \) is of local bounded variation (from instance from the form of the applied forces, or from energetical arguments). Using the tools in [122] theorem 4.2, [538] theorem 3, [414] theorem 2.2, it is then possible to study existence and uniqueness of solutions (which are here to be considered as attainable or reachable sets) of the dynamical problem. Let us finally insist on the fact that existence and uniqueness results do not rely on explicit knowledge of \( h \), but only on its (crucial) property of being of local bounded variation. This in turn should be proved before using the above coordinate change, and is a weakness of this method. Another formulation that could handle the problem in one shot has to be considered. This should be possible with the sweeping process formulation that we describe in chapter 5.

Remark 1.12 The above change of coordinates that allows us to transform a MDE into a Carathéodory ODE is quite similar to generalized state vector transformations in linear systems theory [252], which allow to write a state space representation without derivatives of the input \( u(t) \), for systems with polynomial representation \( A(D)y(t) = B(D)u(t) \). \( D = \frac{d}{dt}, A(D) \) and \( B(D) \) are polynomials of \( D \), with orders \( n \) and \( m \) respectively. Such transformations are of the form \( z = Tyₚ + Tuₚu \), where \( yₚ^T = (y, \dot{y}, \ddot{y}, ..., y^{(n-1)}) \). \( T \) and \( Tu \) are constant matrices of suitable dimensions. It is clear that if a discontinuous control \( u(t) \) is applied, then \( B(D)u(t) \) contains the Dirac distribution and its derivatives up to order \( m - 1 \). In a suitable state space representation (controllable canonical form for instance) such singular terms are absent. Let us come back on a simple nonlinear system as in (1.22). Let us prove that there exists a change of coordinates of the form \( z = Z(x, u) \) such that at
least locally the system becomes in new coordinates \( \dot{z} = h(z, u) \). Indeed one gets

\[ \dot{z} = \frac{\partial Z}{\partial x} \dot{x} + \frac{\partial Z}{\partial u} \dot{u} = \frac{\partial Z}{\partial x} [f(x) + g(x)\dot{u}] + \frac{\partial Z}{\partial u} \dot{u} \]  

(1.51)

A sufficient condition for the transformed system to be in the required form is thus that

\[ \frac{\partial Z}{\partial x} g(x) = -\frac{\partial Z}{\partial u} \]  

(1.52)

Then if we can express \( x = X(z, u) \), i.e. in a sense invert the coordinate change, then we get \( \frac{\partial Z}{\partial x} f(x) = h(z, u) \). Now let us search for a solution to (1.52) of the form \( Z(x, u) = a(x)b(u) \). We obtain

\[ \frac{da}{dx}b(u)g(x) = -a(x)\frac{db}{du} \]  

(1.53)

which we can rewrite as

\[ \frac{1}{a(x)}\frac{da}{dx}g(x) = -\frac{1}{b(u)}\frac{db}{du} \]  

(1.54)

Now note that since each side of the equality must be verified for all \( x \) and \( u \), and since the left hand side is a function of \( x \) while the right hand side is a function of \( u \), both sides must be equal to the same constant value. Therefore we can search for \( a(x) \) and \( b(u) \) such that

\[ \frac{da}{dx} = a(x) \frac{g(x)}{g(x)} \]  

(1.55)

and

\[ \frac{db}{du} = -b(u) \]  

(1.56)

provided \( g(x) \) does not go through zero, which is necessary for the controllability of the system. Then from (1.56) and classical existence of solutions to ordinary differential equations, there is clearly a local solution to the problem. Note however that the resulting system may not be in general linear in the control input \( u \). As an example let us consider the system

\[ \dot{x} = \sin(x) - \cos(x)\dot{u} \]  

(1.57)

(This is inspired from a cart-pendulum system whose complete dynamical equations are of higher order, but with the same nonlinearities). Then the coordinate change \( z = |\tan(\frac{x}{2} + \frac{\pi}{4})|\exp(u) \) transforms the system into

\[ \dot{z} = \frac{z^2 \exp(-u) - \exp(u)}{2} \]  

(1.58)

Notice that the fixed point of (1.57) is given when \( u \equiv 0 \) by \( x = 0 \) and corresponds for (1.58) to \( z = 1 \). However it is possible that other change of coordinates yield a system linear in the input.
1.4.2 From measure to Filippov’s differential equations: the Zhuravlev-Ivanov method

A method to eliminate discontinuities and impulsive forces from the dynamical equations of mechanical systems with unilateral constraints has been proposed in [617] [616] [618] [615], and extended in [230] [235] for the analytical study of vibro-impact systems. Let us first describe the pioneering work of Zhuravlev [615]. A n-degree-of-freedom system with generalized coordinates \( q = (q_1, \cdots, q_n)^T \), Lagrangian function \( L(t, q, \dot{q}) = T(q, \dot{q}) - U(t, q) \) and a codimension one constraint \( q_1 \geq 0 \) is considered. The Routh’s function \( R \) is introduced, as \(^{(25)}\)

\[
R = L(q, \dot{q}, t) - p_y^T \dot{y}
\]  

(1.59)

where \( y \triangleq (q_2, \cdots, q_n)^T \), and \( p = \frac{\partial L}{\partial \dot{q}} (\frac{\partial L}{\partial q_i}, 2 \leq i \leq n, \) are the generalized momenta associated to velocities \( q_i \). Then the nonsmooth transformation

\[
q_1 = |s|
\]  

(1.60)

is introduced. It clearly appears from (1.60) that the basic idea is to consider the “mirror” system such that on \([t_{2k}, t_{2k+1}]\) the fictitious trajectory \( s(t) \) is symmetrical to the actual one \( q_1(t) \) with respect to the constraint surface \( q_1 = 0 \) (The first impact occurs at \( t_0 \)). Notice at once that since \( q_1(t_k) = s(t_k) = 0 \), we obtain

\[
q_1(t) = \frac{d}{dt} \{ s \operatorname{sgn}(s) \} + \sum_{k\geq 0} \sigma_s \delta_{s, \operatorname{sgn}(s)}(t_k) \delta_{t_k} = \dot{s} \operatorname{sgn}(s). \]

It is possible to show that in the coordinates \((s, q_2, \cdots, q_n)\), the dynamical equations become (see also remark 1.13)

\[
\frac{d}{dt} \frac{\partial R_0}{\partial s} - \frac{\partial R_0}{\partial \dot{s}} = \left[ S - \frac{d}{dt} \left( p^T Ab \right) + \dot{s} p^T \frac{\partial Ab}{\partial x} \right] \operatorname{sgn}(s) \quad \text{(1.61)}
\]

\[
\dot{y} = -\frac{\partial R_0}{\partial p} - \dot{x} Ab \operatorname{sgn}(s)
\]

(1.62)

\[
\dot{p} = \frac{\partial R_0}{\partial y} + \dot{x} \frac{\partial p^T Ab}{\partial y} \operatorname{sgn}(s) + Y
\]

where \( S \in \mathbb{R} \) and \( Y \in \mathbb{R}^{n-1} \) are the generalized forces corresponding to \( s \) and \( y \) respectively, the inertia matrix is expressed as \( M(q) = \begin{bmatrix} a & b^T \\ b & A^{-1} \end{bmatrix} \), and \( R_0 = \frac{1}{2} (a - b^T Ab) s^2 - \frac{1}{2} p^T Ap + U(t, q) \). Zhuravlev uses the Routh function as a descriptive function to write down the dynamics precisely because it allows one to avoid the impulses in the final equations. Note that the dynamics in (1.61) have a Lagrangian

\(^{25}\)The Routh’s function is usually introduced for \( n \)-degree-of-freedom systems that possess \( n_c \) cyclic coordinates \([529]\) (i.e. coordinates \( q_1, \cdots, q_{n_c} \) that do not appear in the Lagrangian function \( L \), nor in the Hamiltonian function \( H \)). Every cyclic coordinate yields a first integral of the system since the corresponding momenta \( p_1, \cdots, p_{n_c} \) are invariant. Routh’s method consists of applying a Legendre transformation \([16]\) §14, only in the coordinates \( q_1, \cdots, q_{n_c} \), i.e. the Routh’s function is equal to \( R = \sum_{i=1}^{n_c} p_i q_i - L \). The interest of the Routh’s function is that it plays the role of a Hamiltonian function for the cyclic coordinates, i.e. \( \dot{p}_i = -\frac{\partial R}{\partial q_i} \) and \( \dot{q}_i = \frac{\partial R}{\partial p_i} \).
1.4. CHANGES OF COORDINATES IN MDE

form, whereas the ones in (1.62) have a Hamiltonian form. Most importantly, notice that from (1.61) and (1.62) the state of the transformed system is time-continuous. In particular since \( q_1 = \hat{s} \text{ sgn}(s) \), this implies that \( q_1(t) = s^2(t) \). Hence \( q_1(t) \) is time-continuous. From this fact and continuity of the rest of the variables, it is also possible to show that \( T(t_k^+) = T(t_k^-) \). Hence only lossless shocks are considered.

Example 1.7 To illustrate the method, let us consider the classical example of a linear oscillator constrained by a rigid obstacle. The dynamical equations are given by:

\[
\ddot{q} + \lambda_1 \dot{q} + \lambda_2 q = A \sin(\omega t), \quad q \leq 0
\]

(1.63)

We assume that there is no kinetic energy loss at impacts, i.e. \( q^2(t) \) is time continuous. From (1.60) we obtain \( \dot{s} = \ddot{q} \text{ sgn}(s) \) (indeed \( \dot{s}(t^-) = \dot{s}(t^+) = \dot{q}(t_k^-) \), so that \( \sigma_{\text{sgn}(s)}(t_k) = \sigma_q(t_k) \)). Therefore we get

\[
\dot{s} + \lambda_1 \dot{s} + \lambda_2 s = A \sin(\omega t) \text{ sgn}(s)
\]

(1.64)

Note that it is not clear whether such transformation can be performed when the system is subject to several constraints, like for instance an inverted pendulum between two walls.

Ivanov [235] extends the method to non purely elastic shocks. He first considers the one degree-of-freedom system

\[
\begin{align*}
\ddot{x} &= f(x, \dot{x}, t) & \text{if } x \geq 0 \\
\ddot{x} &= \max(0, f) & \text{if } \dot{x}(t_k^+) = 0 \text{ and } x(t_k) = 0 \\
\dot{x}(t_k^-) &= -e \dot{x}(t_k^-) & \text{if } \dot{x}(t_k^-) < 0 \text{ and } x(t_k) = 0
\end{align*}
\]

(1.65)

where \( x \in \mathbb{R} \). As we shall see later, the last algebraic equation in (1.65) is a restitution law that is in fact needed to render the dynamical problem complete, i.e. the postimpact values can be computed from the preimpact ones. Note also that the second condition in (1.65) is necessary to prevent any penetration into the constraint \( x \geq 0 \). Indeed assume for instance that at the impact time \( t_0 \), we have \( f(x, \dot{x}, t) < 0 \): then if \( e = 0 \), \( \dot{x}(t_0^+) = 0 \). Now integrating the smooth motion dynamics, one would find that \( x < 0 \) for some \( t > t_0 \) (This is the case if \( f(x, \dot{x}, t) = -g \): then one finds \( x(t) = -\frac{1}{2}(t - t_0)^2 \)). But if in this case one considers the second equation, then it follows that \( \dot{x}(t_0^+) = 0 \), and \( x(t) = 0 \) for all \( t \geq t_0^+ \). The system remains stuck on the surface after the first impact.

It can be shown that (1.65) can be transformed to:

\[
\begin{align*}
\dot{s} &= R \nu \\
\dot{\nu} &= R^{-1} \text{sgn}(s)f(t, |s|, R\nu \text{ sgn}(s))
\end{align*}
\]

(1.66)

26Such a model has been extensively used in the literature to study the dynamics of mechanical devices subject to collisions, see e.g. [417] [199] [191] [257] [258] [237] [335] [336] [339] [617] [238] [239] [240] [338] [339] [340] [401] [310] [236] [262] [483] [370] [401] [431] [581] [484].
Once again let us reiterate that trajectories of the system in (1.66) are defined as time-continuous functions, see remark 1.15. At the origin \((s, \nu) = (0, 0)\), the transformed system is defined as \(\dot{s} = 0\), \(\dot{\nu} = (1 - k)^{-1} \max(0, f(t, 0, 0))\). The nonsmooth coordinates change is given by:

\[
\begin{cases}
\begin{align*}
\dot{x} &= |s| \\
R &= 1 - k \text{sgn}(s) \nu
\end{align*}
\end{cases}
\]

From \(x = |s| = s \text{sgn}(s)\) one deduces that \(\dot{x} = \frac{d}{dt} \{s(t) \text{sgn}(s(t))\} + \sigma_x(t_j) \delta_{t_j}\), where \(t_j\) denotes generically an instant such that the sign of \(s(t)\) changes. From the fact that \(\sigma_x(t_j) = s(t_j^+) \text{sgn}(s(t_j^+)) - s(t_j^-) \text{sgn}(s(t_j^-))\) and since \(s(t)\) is continuous (because we know that \(x(t)\) is), it follows that \(\dot{x} = s \text{sgn}(s) = R\nu \text{sgn}(s)\) by (1.67). Hence the first equation in (1.66). Concerning the acceleration, one gets \(\ddot{x} = f(t, x, \dot{x}) + \sigma_{xx}(t_k) \delta_{t_k} = \frac{d}{dt} \{R\nu \text{sgn}(s)\} + \sigma_{x\nu \text{sgn}(s)}(t_j) \delta_{t_j}\), where \(t_j\) denotes generically an instant where \(\dot{x} = R\nu \text{sgn}(s)\) may be discontinuous: inspection of (1.67) shows that this can occur when \(s(t)\) or \(\nu(t)\) cross zero. If \(\nu(t_j) = 0\), then it is clear that \(\sigma_{xx}(t_j) = 0\). Hence \(\ddot{x} = R\nu \text{sgn}(s)\) so that \(\dot{\nu}\) is given by the second equation in (1.66). But if the trajectory intersects the \(\nu\)-axis with \(\nu \neq 0\), then the change of sign is due to \(s\) and \(\sigma_{xx}(t_j) = 2\nu(t_j) \text{sgn}(s(t_j^+)) = \sigma_{xx}(t_k)\), where \(t_k\) and \(t_j\) coincide. Hence starting from the transformed system in (1.66), we retrieve that if \((s, \nu)\)-trajectory crosses the \(s\)-axis, no impact occurs. If it crosses the \(\nu\)-axis, then this occurs when \(s = 0\) (i.e. \(x = 0\), the constraint is attained), and an impact occurs since \(\sigma_{xx} \neq 0\) from (1.67).

**Remark 1.13** One easily realizes that the coordinate change is not invertible \((x\) as a function of \(s\) is not a monotonic function). Given any couple \((s, \nu)\) one can compute \((x, \dot{x})\), but the inverse implies knowledge of the dynamics, i.e. the collisions times \(t_k\). Thus the method should be considered as a trick allowing to define the original system from the transformed one (This is always possible starting from (1.66) and then computing \(x\) and \(\dot{x}\) from (1.67)). It is more accurate to say that one starts from the transformed system in (1.66), and that the coordinate change in (1.67) allows to retrieve the original dynamics.

From (1.67) one sees that for plastic impacts \((e = 0)\) the transformation is not well-defined, since \(R = 0\) when \(\text{sgn}(s) = \text{sgn}(\nu)\). Then a trajectory that attains the \(s = 0\) axis instantaneously reaches the equilibrium point \((s, \nu) = (0, 0)\). Whatever the coordinate change may be, this fact is invariant since for a plastic impact, the equilibrium (rest) position is attained immediately after the impact, which corresponds in the \((s, \nu)\)-plane to intersecting \(s = 0\). In addition \((s, \nu) = (0, 0)\) implies \((x, \dot{x}) = (0, 0)\) and \(x = 0\) implies \(s = 0\); now note that at \(s = x = 0\), \(\dot{x}\) is not defined, but the right and left limits when \(s \to 0\), \(s > 0\) or \(s < 0\) respectively, are defined. We retrieve here the discontinuity in the velocity at impact times. The trajectories of (1.66) are to be understood in the sense of Filipov [154], and are time-continuous, see remark 1.15. Uniqueness of solutions of (1.66) fails if \(s(t_j) = \nu(t_j) = 0\).
1.4. CHANGES OF COORDINATES IN MDE

In fact the basic idea behind the coordinate change in (1.67) is to find out a function $F(s, \nu)$ such that

$$F(0^+, \nu) = -eF(0^-, \nu) \quad \text{for} \quad \nu > 0$$

$$F(0^-, \nu) = -eF(0^+, \nu) \quad \text{for} \quad \nu < 0$$

Then the coordinate change is defined as $x = |s|$ and $\dot{x} = F(s, \nu)$. It is possible
to define other transformations, discontinuous in $s = 0$ only [235]. Such another
possible function $F(s, \nu)$ is given by [235]

$$F(s, \nu) = \left(1 - 2\frac{R}{\pi} \arctan \left(\frac{\nu}{s}\right)\right) \nu \text{sgn}(s)$$

Then the transformed system vector field has discontinuities only on the axis $s = 0$.

Additional comments and studies

Let us note that contrarily to the method presented in subsection 1.4.1 (see (1.50)),
the "impact function" $h(t)$ does not appear in this coordinate change, so that no
time-dependence is added to the transformed system. This is at the price however
of obtaining a state-discontinuous vector-field in (1.66), the surface of discontinuity
corresponding to the surface of constraint $x = 0$. The advantage of the method in
(1.50) is that the resulting system is Carathéodory. A drawback is that although
explicit knowledge of $h$ is not needed for existence and uniqueness investigations,
it might be needed for stability purposes (Moreover recall that it is not obvious to
prove that $h(t)$ is $RCLBV$). Therefore it seems that the form in (1.66) is much
more suited for such analysis: some local stability analysis based on linearization
are led in [235]. Smooth [154] or nonsmooth [489] generalized Lyapunov functions
could also be used in this setting.

Ivanov [235] studies singular points, stability of equilibriums, stability of peri-
odic motions and bifurcations in vibro-impact systems using this setting. Finally the
techniques can be extended to $n$-dimensional systems with a single constraint, and
Coulomb friction can be considered. The work in [172] has been inspired by Zhu-
ravlev [615] to study the motion of a simple mechanical system with clearance and
impacting via a nonsmooth coordinate change, and an averaging method to study
periodic motions. Analytical, numerical and experimental results are presented in
[172] and are in accordance. Numerical investigations in [349] show that the nons-
smooth transformed system of Ivanov can be successfully applied to investigate the
dynamics of mechanical systems with unilateral constraints, using software pack-
ages developed for continuous or discrete-time systems (e.g. the software INSITE
is used in [349]). This is of great practical importance since numerical calculations
are often the only way to study the dynamics of complicated systems. No software
seems to exist that allow to directly treat such hybrid systems with both continuous
(between impacts) and discrete (at impacts) dynamics. Ivanov's transformation is
thus an important step in the analysis of such systems.
Remark 1.14 In relationship with remarks (1.3.2) and (1.5), notice that it clearly appears from the transformed dynamics in (1.50) or (1.66) that autonomous systems with unilateral constraints are still autonomous. Indeed \( h(t) \) in the right-hand-side of (1.50) is a step function whose jumps at the collision times depend only on the system's state (preimpact velocity), via a restitution rule. Also it is obvious that the right-hand-side of (1.66) does not explicitly depend on time as long as the external action \( f(x, \dot{x}) \) does not.

Remark 1.15 A thorough description and analysis of solutions of systems with right-hand-side discontinuous in the state variable can be found in Filippov's book [154]. The solutions of a system like the one in (1.66) are defined from a differential inclusion point of view, as absolutely continuous functions. Conditions for existence and continuity with respect to initial conditions are given [154] §7 and 8, chapter 2. Roughly, systems like in (1.66) are replaced by a differential inclusion, whose right-hand-side (a multivalued function) is either a point outside the discontinuities, or a line segment at discontinuities, joining the left and right limits of the vector-field. Provided the set hence defined verifies some basic properties (nonemptyness, boundedness, closedness and convexity) and provided some upper semicontinuity conditions for the multivalued function are true, existence, uniqueness and continuous dependence on the initial data are guaranteed, [154] §8, chapter 2, corollary 3. Other results closer in spirit to those for ODE's can be found in [588]. Also discontinuous measurable in \( t \) external forces \( f(t, x, \dot{x}) \) may be treated via the results in [414].

Remark 1.16 The nonsmooth Zhuravlev-Ivanov coordinate change is limited to codimension one constraints (several constraints may be considered, but then the variable change is valid locally only in the neighborhood of one of the constraints). An important particular case when several constraints may be considered is when the constraints surfaces are mutually orthogonal in the kinetic metric. Further details on such cases are given in chapter 6.

Remark 1.17 The transformed system will generally involve the product of the input by a term containing \( \text{sgn}(s(t)) \). When \( s(t) \) crosses the \( \nu \)-axis, then \( \text{sgn}(s(t)) \) is a time discontinuous function. Hence one retrieves that an impulsive input cannot be applied at such times (that correspond to impact times), otherwise yielding a non-wellposed system.
Chapter 2

Approximating problems

Clearly, the assumption that all bodies are perfectly rigid, hence the introduction of singular distributions in the modeling of collisions has the advantage of providing a simple and systematic derivation of impact dynamics. Note however that bodies that collide may possess a certain compliance, so that the collision duration is strictly positive and local deformations occur near the point of impact. In that sense, rigid bodies dynamics may be considered as a limit case only, which however does not preclude its practical as well as theoretical utility. This is the reason why many researchers (see among others [174, 243, 244, 249, 305, 359, 614]) have chosen to work with continuous-dynamics models of collision, such that the bodies deform during the impact, and the collision dynamics are treated as continuous time dynamic phenomenons. The models used may be quite complex and rely on advanced mechanical engineering tools like finite elements methods, Hertz’s or Cattaneo-Mindlin’s theories of impact, see [243] [244] for details. As noted in the introduction, we will not describe here these models, since we rather focus on the use of compliant models to better analyze and understand the rigid case. The (linear) models used are generally based on combinations of masses, springs and dampers (the spring-dashpot model). It has been argued [216] [132] that such models for the contacting surfaces are well-suited, because the energy-loss at impacts is associated primarily with damping rather than micro-plastic deformation or permanent strain. These conclusions are more or less confirmed in [598]. It is pointed out in [598] after numerical and experimental investigations on impacts of a flexible arm against a rigid obstacle that although a spring-dashpot and a more sophisticated Hertzian-like model with additional plastic effects provide similar results, the spring-dashpot parameters are more difficult to identify.

Also, it is well-know that perfect rigidity may yield in certain cases undetermined systems (i.e. with no or several solutions). We shall see some of these problems in more details in chapter 5, section 5.4. These undeterminancies can be avoided by adding some compliance in the system. This is the classical case of hyperstatic mechanisms in mechanism theory.

On the other hand, it is a current procedure in mathematics to study a given
problem $\mathcal{P}$ (nonsmooth dynamical systems) by considering it as the limit (in a sense to be precised) of a sequence of problems $\mathcal{P}_n$ (smooth dynamical systems) with known strong properties (existence, uniqueness, continuous dependence on parameters...). Furthermore, we shall see that this path once again enables us to conclude about the distributional nature of solutions of rigid bodies impact problems. Let us note however the following: the point of view that consists of associating a sequence of compliant problems to a rigid limit problem is not the unique one one may have on impact dynamics. Similarly as one can view distributions either as functionals or as limits of sequences of functions (see appendix), one can consider nonsmooth percussion dynamics directly as measure differential equations with no reference to any elastic environment. Of course it should logically be possible to associate physically realizable compliant models to any rigid problem, but as we shall see in the sequel this is not always an obvious task. Note also that sometimes the consideration of compliant problems sequences is a very natural way to prove some compactness results, for instance in minimization problems (see the section on Lagrange equations and Hamilton's principle). This is at the core of some mathematical results for variational calculations, like the $\Gamma$-convergence [118].

2.1 Simple examples

In this section we study the relationships between impulsive and physical models of percussions in a simple case, i.e. we show that the interaction force converges towards a Dirac measure when the stiffness and/or the damping coefficient grow unbounded.

2.1.1 From elastic to hard impact

Consider for example that we now attach a spring of stiffness $k$ to the mass $m$ moving on a horizontal line, and that the mass collides with a wall (infinite mass) through the spring, at time $t_0$ (the system is depicted in figure 2.1). The dynamic
2.1. SIMPLE EXAMPLES

The equations of this simple system are:

\[
0 \leq t < t_0 \quad m \ddot{x} = 0, \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0
\]

\[
t_0 \leq t < t_1 \quad m \ddot{x} + kx = kx(t_0)
\]

where the spring remains in contact with the wall on \([t_0, t_1]\). Assume for simplicity that \(x(t_0) = 0\) and \(t_0 = 0\), i.e. the percussion instant and coordinate are chosen as origins. We obtain in the new translated coordinates \(x(t) = x_0 \sqrt{\frac{m}{k}} \sin (\sqrt{\frac{k}{m}} t)\). The first time instant after the impact such that \(x(t_1) = 0\) is \(t_1 = \sqrt{\frac{m}{k}} \pi\) (i.e. the spring is being crushed and then restores its potential energy). Note that \(\dot{x}(t_1) = -\dot{x}_0\).

Consider now any sequence of stiffness values \(\{k_n\}, \quad n \in \mathbb{N}, \quad k_n < k_{n+1}, \quad k_n \to +\infty\) as \(n \to +\infty\). Let us denote \(p_n(\tau) = k_n x_n(\tau)\) for \(0 \leq \tau < t_1\), \(p_n(\tau) = 0\) elsewhere. Note that the subscript \(n\) in \(x_n(t)\) is to emphasize that \(x(t)\) is a solution of an approximating problem. Then \(\int_0^{t_1} p_n(\tau) d\tau = \int_0^{t_1} k_n x(\tau) d\tau = 2x_0 m\) for all \(n > 0\).

Now notice that \(x(t)\Delta x_n(\tau) \to 0\) on \([0, t_1]\) as \(n \to +\infty\) and \(t_1 \to 0^+\) as \(n \to +\infty\), i.e. if the stiffness is infinite, \(x\) remains unchanged during the impact and the impact duration is zero. Moreover the compliant elastic collision tends towards a hard collision. It is easy to verify that the sequence \(p_n(\cdot)\) of contact force functions converges to \(2k_0 x_0\) as \(n \to +\infty\), by checking conditions i, ii and iii for delta-sequences given in appendix A, section A.2.

Following the terminology in [93] [84] [85] [427] [428] we have chosen a penalizing function \(\psi_n(x_n) = -k_n x_n\) if \(x_n > 0\), 0 if \(x_n \leq 0\), that exactly fits within the conditions imposed by these authors, see e.g. [84] equation (0.1), and section 2.2 in this chapter.

**Remark 2.1** In this case the impulse during the compression phase is \(P_c = -m \dot{x}(0)\) and during the expansion phase \(P_e = -m \dot{x}(0)\). This suggests that the Poisson's rule may also be given a signification when \(k \to +\infty\) by defining \(P_c = -m \dot{x}(0^-)\) and \(P_e = m \dot{x}(0^+)\).

**The work performed by contact forces**

The work effectuated by the contact force during the impact is given by \(W_{[0,t_1]} = \int_0^{t_1} \dot{x}(t) k_n x(t) dt = \frac{m x_0^2}{2} [\cos^2(\pi) - 1] = 0\) for any \(k_n \in \mathbb{R}^+.\) Thus it seems reasonable to consider that the work of the impulsive force at the impact time is zero, although, as we pointed out in the preceding section, the product \(\dot{x} \delta_e\) is not defined. This is consistent with the lossless property of this model. It may be expected that different approximating models yield different limit values of this work. We should however restrict ourselves to physical models, i.e. models that do possess a "mechanical structure".

Let us note that the maximum interaction force value is given by \(F_{\text{max}} = \dot{x}_0 \sqrt{km}\) and therefore tends to infinity as \(k\) grows unbounded. Thus the intuitive and widely spread idea of "very large" forces at impacts seems quite justified from this physical
example. We however think that this is a wrong way to interpret the example. In fact, we tend to believe that the right idea is still to consider the interaction force $F(\cdot)$ as a distribution in $D^*$, so that one easily sees that the "effect" of $F$ on any function in $D$ with support containing $0$ is finite, for any $k$, because the support of $F$ tends towards zero so that $F$ becomes atomic. As we have already seen, what is to be considered as the impact magnitude for infinitely large $k$ is the magnitude $p_k$ of the impulse, which exactly corresponds to the integral of the interaction force over the contact interval (i.e. what is called the impulse of the interaction force during the contact period), and not to the maximum value of this force that makes no sense when $k = +\infty$. However for practical purposes one may also argue that the maximum value of the interaction force is important (to be able to prevent possible damage of the materials in contact) [598]. Then it is clear that a rigid body model cannot predict such value. One must then use a compliant approximating model of the contact-impact process.

2.1.2 From damped to plastic impact

Instead of a spring, let us now attach to the mass a damper with coefficient $f$, as shown in figure 2.2. The dynamic equations are:

\begin{align*}
0 \leq t \leq t_c & \quad m\ddot{x} = 0, \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0 \\
t_c \leq t \leq t_1 & \quad m\ddot{x} + f\dot{x} = 0
\end{align*}

(2.2)

Proceeding as above, we obtain after the impact $\dot{x} = \dot{x}_0 e^{-\frac{f}{m}t}$, $x = -\frac{m\ddot{x}_0}{f} (1 - e^{-\frac{f}{m}t})$.

One sees that if $f \to +\infty$, then $x(t) \to x(0) = 0$ and $\dot{x} \to 0$ for all $t > 0$. For any sequence of values of damping coefficient $\{f_n\}$ defined as in the preceding example, let us denote $p_n(\tau) = f_n\ddot{x}_n(\tau) + m\dot{x}_0\sqrt{\frac{f_n}{m}}e^{-\sqrt{\frac{f_n}{m}}\tau}$ for $0 \leq \tau \leq \sqrt{\frac{m}{f_n}}$, $p_n(\tau) \equiv 0$ elsewhere. Then $\int_0^{\sqrt{\frac{m}{f_n}}} p_n(\tau) d\tau = \int_0^{+\infty} f_n\dot{x} d\tau$. Note that this time the interaction impulse is calculated on the whole interval $[0, +\infty)$ since the body never detaches from the surface. It is easy to check that $p_n(\cdot)$ verifies conditions i, ii and iii in appendix A. Hence we get $p_n(\cdot) \to m\dot{x}_0\delta_0$ as $n \to +\infty$. In the limit, the equation describing the system becomes $m\ddot{x} = (p - m\sigma_x(0))\delta_0$, with $\sigma_x(0) = \dot{x}(0^-) = \dot{x}_0$. 

Figure 2.2: Plastic compliant contact-impact model.
2.1. SIMPLE EXAMPLES

2.1.3 The general case

Consider now that we attach a spring \( k \) and a damper \( f \) to the mass as in figure 2.3 (1). During the contact we get \( m\ddot{x} + f\dot{x} + kx = 0 \). Assume now that \( \Delta = f^2 - 4km < 0 \). Then we obtain \( x(t) = \frac{x_0}{\omega} e^{rt} \sin(\omega t), \dot{x}(t) = \frac{x_0}{\omega} e^{rt} \left[ \frac{r}{\omega} \sin(\omega t) + \cos(\omega t) \right] \), with \( x_0 = \dot{x}(0), r = \frac{f}{2m}, \omega = \frac{\sqrt{-\Delta}}{2m} \). Now for \( t_1 = \frac{r}{\omega}, x(t_1) = 0 \) and \( \dot{x}(t_1) = -x_0 e^{\frac{r}{\omega}} \). Let us choose \( 0 < \beta \leq 1 \), and let us see what happens if (2)

\[
f = 2|\ln(\beta)| \left( \frac{km}{\pi^2 + \ln^2(\beta)} \right)^{\frac{1}{2}}
\]

when \( k \to +\infty \) (Such an \( f \) guarantees \( \Delta < 0 \) for \( 0 < \beta \leq 1 \)): we get \( t_1 \to 0 \) and \( e^{\frac{r}{\omega}} \to \beta \). Thus \( \dot{x}(t_1) \to -\beta x_0 \) as \( k \to +\infty \) (if \( \beta = 1 \) then \( f \equiv 0 \) and we retrieve the above case, and if \( 0 < \beta < 1 \), then \( f \to \infty \) as \( k \to \infty \)). Simple calculations show that

\[
\int_0^\pi p_n(\tau) d\tau \Delta \int_0^\pi (f_n \dot{x}_n(\tau) + k_n x_n(\tau)) d\tau = m\dot{x}_0 (\beta + 1), \text{ where } \{k_n\} \text{ is a sequence of stiffness coefficients defined as previously, and } f_n = 2|\ln(\beta)| \left( \frac{k_0 m}{\pi^2 + \ln^2(\beta)} \right)^{\frac{1}{2}}.
\]

Thus once again the sequence of force functions \( p_n(\cdot) \) during the collision time converges towards a Dirac distribution. Note that to show this, we have considered a sequence of damping coefficients that depends on the mass \( m \). This is at first sight surprising, as one can expect the nature of the collision to be independent of the mass of the bodies that collide. However what really matters is not how the sequences \( \{f_n\} \) and \( \{k_n\} \) are defined but rather that they do exist, i.e. we are able to associate a sequence of compliant models to the rigid limiting model.

Remark 2.2 Note that when \( \beta \to 0, \beta > 0 \), then \( \Delta \to 0, \Delta < 0, f \to 2\sqrt{km} \), and as \( k \to +\infty \) (the sequence \( \{k_n\} \) can be chosen of the form \( k_n = \ln^2 \beta k'_n \), with \( k'_n \to +\infty \)) then \( t_1 \to 0 \) and \( \dot{x}(t_1) \to 0 \) also. More formally, let us define a sequence of positive coefficients \( \beta_j \), with \( \beta_j \to 0 \) as \( j \to +\infty \). Hence we define

\footnote{This model is often called a spring-dashpot model.}

\footnote{Let us note that the following relationship means that the damping coefficient is taken to be proportional to the square root of the stiffness coefficient. The approximating problems in [344] are chosen such that the mass varies as the square of the damping. Although the approximating sequences in [344] and in this chapter are constructed for quite different purposes (see chapter 1 section 1.2 for some details on [344]), it is worth noting that the various mechanical coefficients vary at different rates in both cases.}
CHAPTER 2. APPROXIMATING PROBLEMS

the functions \( p_{n,j}(t) = f_{n,j} \dot{x}_{n,j}(t) + k_{n,j} x_{n,j}(t) \). Then from the above it follows that \( p_{n,j}(\cdot) \to m \ddot{x}_0 (\beta_j + 1) \delta_0 = p_j \delta_0 \) and trivially \( p_j \delta_0 \to m \ddot{x}_0 \delta_0 \) as \( j \to +\infty \) (Convergence is always understood in the sense of distributions, see appendix A, section A.3). Therefore \( p_{n,j}(\cdot) \to m \ddot{x}_0 \delta_0 \) as \( n \) and \( j \to +\infty \). We have thus found out two different sequences of interaction forces, both based on simple mechanical models of contact-impact, that both approximate the same limit problem, i.e. a purely inelastic shock. Later we shall see that we can associate to rigid impacts problems a restitution coefficient denoted generally as \( e \): elastic impacts correspond to \( e = 1 \), purely inelastic percussions correspond to \( e = 0 \), and in general \( e \in [0, 1] \).

Here we had in fact \( e = \beta \). It is worth noting that there is a fundamental difference between the cases of purely dissipative shocks (\( e = 0 \)) and elastic shocks (\( 0 < e \leq 1 \)): indeed in the first case the impact corresponds to an instantaneous dissipation of the whole energy of the system, but the compliant model does not necessarily tend towards an infinitely rigid model; on the contrary when a spring is added in the compliant model then the limit is an infinitely rigid surface. Thus the two different models that both converge towards a purely dissipative shock may be considered quite different from a physical point of view. Another problem is to decide which one better fits with a given practical problem.

The work performed by contact forces

The work performed by the contact forces during the impact is given this time by \( W_{[0,t]} = \int_{t_0}^{t_1} \dot{x}(t)(f_n \dot{x}(t) + k_n x(t))dt = \frac{1}{2} \frac{k_n^2}{2 \omega^2 + \pi^2} \left[ e^{\frac{\pi \omega}{\omega^2 + \pi^2}} - 1 \right] \) that tends towards \( \frac{m \dot{x}(0)^2}{2} (\beta^2 - 1) < 0 \) when \( k \to +\infty \). Note that this last value is exactly the loss of kinetic energy \( T_L \) at impact and can also be deduced from the result in [536] that states that the work performed by the impact of a particle against a massive barrier is given by an "average" formula \( W_{[0,t]} = P_n(t_1) \frac{\dot{x}_n(t_1) + \dot{x}_n(0)}{2} \) where \( P_n(\cdot) \) is the normal impulse of the percussion, i.e. the time integral of the interaction force during the shock interval (thus \( P_n(0) = 0 \)) \(^3\). \(^4\) The formula in [536] has been extended to the case when there is unidirectional slip during the impact in [521], where it is shown that in this case \( W_{[0,t]} = (P_n(t_1) - P_n(0)) \frac{\dot{x}_n(t_1) + \dot{x}_n(0)}{2} \). Thus once again the approximating model allows us to give a value to the impulsive work, that is consistent with the energetical behaviour of the impact (here a loss of energy as long as \( \beta < 1 \)). Note that the distributional formulation allows to easily calculate the loss of kinetic energy but not the impulsive forces work. This is somewhat paradoxical since both quantities are equal and represent the same physical process of energy dissipation. The work performed by impulsive forces is sometimes [36] [222] [506] [502] [519] deduced from (1.1) as \( W_i = \lim_{\Delta t \to 0} \int_{t_0}^{t_1 + \Delta t} p(\tau) \dot{x}(\tau)d\tau \): it is clear that without any approximating sequence of impact problems, this is meaningless in the distributional sense as we have already pointed out.

\(^3\)This is not valid in general when friction is present with slipping [36] [502].

\(^4\)Thompson and Tait's result is known as Kelvin's formula [234] when the normal impulses and velocities are replaced by the total tangential + normal impulses and velocities.
2.1. SIMPLE EXAMPLES

We thus have proved the following

**Claim 2.1** Consider the equation in (1.2) that represents the dynamics of a rigid mass colliding a rigid environment, without any external action. Then for any energetical behaviour of the materials at the impacts (namely for any restitution coefficient \(0 \leq e \leq 1\)) we can associate an approximating sequence of compliant models such that the approximating solutions \(x_n\) converge uniformly towards the solution of (1.2).

Roughly, as we pointed out in remark 1.2, we have approximated the limit rigid problem by sequences of differential equations of the form \(m\ddot{x}_n = p_n(t)\) for a given sequence of functions \(\{p_n(t)\}\) whose limit is a Dirac measure. Results for convergence of this kind may be found for instance in [154] chapter 1 (see in particular lemmas 4 and 5 §1, theorem 1, §2). Uniform convergence can be proved by using the change of variables indicated in example 1.1 and example 1.3 since the resulting system is Carathéodory. This trick holds for \(x_n\) only since \(\dot{x}_n\) is continuous and cannot thus converge uniformly to a discontinuous \(\dot{x}\). The result holds for more general systems like the one in example 1.3 as long as the "impact function" \(h(t)\) is of local bounded variation. We have been able to prove the above because the considered problem is integrable and the exit times can be calculated. As pointed out in the previous section, in the general case the problem is much more involved. A possible work is to find out arguments proving that sequences \(\{f_n\}\) and \(\{k_n\}\) exist that yield the same results when for instance an external force \(u(t)\) acts on the system. As an illustration, consider the classical bouncing ball problem, that corresponds to adding a constant force (gravity) to the mass: then it can be shown that the impulse function acting on the ball for \(0 < e < 1\) has the form \(p = \sum_{k=0}^{+\infty} p_k \delta_{t_k}\), where \(t_\infty < +\infty\) is an accumulation point of the sequence \(\{t_k\}\). A fundamental property of this sequence is that the step function \(h(t) = \sum_{k=0}^{n} p_k\) on \([t_k, t_{k+1}]\), \(n \geq 0\) is of bounded variation on \([t_0, t_\infty]\). Hence from [478] p.25 and theorem 2, p.53, \(p = h\) can be considered as a Schwartz's distribution since it is a bounded measure. Then from density of \(\mathcal{D}^*\) in \(\mathcal{D}^*\) [478] theorem 15, chapter 3, we can approximate \(p\) and \(h\) by sequences of smooth functions \(\{p_n\}\) and \(\{h_n\}\). From [12] lemma 2.2.5, \(h_n = p_n\) since the functions \(h_n\) are continuous. Are there sequences of damping and stiffness coefficients such that the corresponding compliant model is an approximating sequence for this problem? The work in [416] that we describe below yields a positive answer.

**Remark 2.3** The transition from compliant to rigid problems can also be attacked *via* singular perturbations techniques: basically, one lets a coefficient of the differential equation be very large, and studies the dynamics by suitably separating so-called fast and slow dynamics. This allows to investigate the behaviour of some

\(^5\)Recall that strictly speaking, this fact has to be proved. In the simple examples we have treated, we have been able to integrate the equations and to calculate the functions \(p_n(\cdot)\). Obviously in slightly more complex cases this would not be possible.
systems, locally in time and in the parameters sets(some conditions guarantee the
global validity in time). This has been investigated e.g. in [359] for the case of a
n-degree-of-freedom of manipulator in contact with a dynamic model of environ-
ment, the main purpose being to facilitate simulation of contact-non contact tasks.
Note that this way of proceeding is severely criticized in [32] [437] [151] since it may
induce wrong conclusions depending on the stiffness value, essentially due to numer-
ical problems for very stiff ordinary differential equations: imagine for instance one
chooses a very large stiffness value to simulate the simple examples treated above.
This will imply calculation of very large values that may introduce approximations
due to very fast dynamics (In the example bove one sees that the contact duration
$[0, t_1]$ is directly proportional to $\frac{1}{k^2}$ and a necessarily strictly positive integration
step. The solution is to choose a smaller $k$, but then will this problem still ap-
proximate accurately enough the rigid one? Certainly not in general. Examples are
treated numerically in [437] and in [151] that show that it is not obvious to use a
compliant approximating problem and draw conclusions on the system's dynamics.
The simulation is thus far from obvious. We shall come back briefly on numerical
solutions for systems with unilateral constraints and impacts in subsection 5.4.5).
Then methods based on nonsmooth mechanics can be preferable for numerical inves-
tigations [437]. Non-conservative impacts have been also considered in [291] [129],
Extending a pioneering work on convergence of smooth problems to rigid ones (for
bilateral holonomic constraints) in [463]. Lagrangians of compliant problems with
potential energy $U_n = U + \frac{1}{2} c^2 n f^2(x_n)$, and dissipation force $F_n = k\sqrt{n f^2(x_n, \dot{x}_n)}$
are used in [291] [129]. Then it is proved using singular perturbations techniques
that the compliant problems converge towards a rigid one with Newton’s rule as an
impact law. Note that we retrieve the fact that the potential term is proportional
to the square root of the elastic one as in our simple example above. Similar cal-
culations of the restitution coefficient as above can be found in [292] chapter 1, §2
and §3, see also [449].

The calculation of restitution coefficients via approximating problems has been
considered in the mechanical engineering literature. For instance the authors in
[595] calculate it for a one degree-of-freedom system composed of a mass related to
the ground by a spring of stiffness $k$, and striking in a compliant obstacle composed
of a spring $k_1$ and a damping coefficient $f$. Then it is found that $e = -\frac{x(t_f)}{x(t_0)} =
\exp\left(\frac{\xi f}{\sqrt{1-\xi^2}}\right)$, where $\xi = \frac{f}{\sqrt{(k+k_1)n}}$. The work in [595] contains no rigorous analysis
of the limit case when $k_1$ and $f$ tend to infinity, but it is worth noting that $e$ depends
on the systems parameters, and has a form similar to the one we have calculated
above for a simple system. In particular it is natural that the coefficient depends on
the mass of the system, just like reduced damping and natural frequency in a second
order mechanical system. But in the rigid limit case this dependence disappears,
since it is obvious from our calculations that the value of the coefficient when the
stiffness tends to infinity does not depend on the mass. Hence dependence on the
system parameters and the initial contact velocity reflects the fact that the bodies

\footnote{Such method is called \textit{genetic} in [292].}
are not perfectly rigid. Considering \textit{a priori} restitution coefficients that depend on the approach velocity and some other parameters can be seen as a way of introducing a correction in the rigid body model, in order to incorporate some compliance \textit{via} the coefficients.

Remark 2.4 As we pointed out in the introduction of this chapter, an active topic of research consists of analyzing the contact-impact problem \textit{via} more sophisticated compliant models. For instance, Brach and Dunn [60] study the collisions of microparticles against a rigid obstacle (where adhesion forces play a significant role in the contact-impact process) with the Hertzian model

\[ m\ddot{x} = -\sqrt{r}kx^3 - \sqrt{r}kx^3c_h\dot{x} + 2\pi af_0 + 2\pi af_0c_a\dot{x} \]  

(2.4)

where $x$ is the normal displacement, $k = \frac{2}{3}\pi(k_1 + k_2)^2$, $k_i = \frac{(1-Ei)}{\pi E}$, $\nu_i$ is the Poisson's ratio (an elastic constant) of the impact process sphere-obstacle, $E_i$ is the modulus of elasticity, $k$ is the Hertzian elastic constant, $f_0$ is the magnitude of the adhesion line force, $r$ is the radius of the sphere, $a$ is the contact radius (i.e. the radius of the contact area, $a^2 = rz$), $c_h$ and $c_a$ are dissipation coefficients. In summary, the first term in the right-hand-side is the classical Hertzian restoring force, the second is a dissipation term, the third one is an idealized adhesion attraction force term, the fourth one accounts for dissipation due to adhesion. Although it is outside our scope to study such models, it is worth being aware that they may be in some applications of great usefulness.

2.2 The method of penalizing functions

From now on, we have started from the nature of the interaction force (for both external percussions as well as unilateral constraints) to treat the impact dynamics as \textit{measure differential equations}, and we have analyzed a possible way to approximate these equations by smooth ones that rely on simple mechanical models (Our motivation for doing this was primarily to confirm mathematically the intuitive idea according to which such convergence holds, and also to provide "logical" values of the work performed by contact forces, a problem that does not seem solvable otherwise). This in fact provides also the basic assumptions to the sweeping-process (see section 5.3), where approximating problems using for instance Yosida's approximants may be used to prove well-posedness of the rigid bodies dynamics [366].

In view of the simple examples treated in the preceding subsection, it is natural to consider still another possible analysis, that does not rely on explicit integration of the motion of the approximating problems (which is impossible in general). The framework is that of functional analysis and variational methods, i.e. study of existence and uniqueness of solutions to second-order differential equations with end-point conditions [71] (7). Roughly speaking, the aim is to prove that some se-

\[7\text{It is interesting to note that some compliant problems studied e.g. in [614] are also formulated in a variational framework, \textit{via} elasticity theory, and may be solved with finite elements procedures.}\]
sequence of second order differential equations $P_{n,f}: \ddot{q}_n + \psi_n(q_n, \dot{q}_n) = f(t, q_n, \dot{q}_n)$, that is considered to represent the physical model of a body colliding with a compliant constraint, converges towards a limit problem in which the constraints are rigid, i.e. unilateral. By convergence it is meant that solutions $q_n(t)$ converge. The functions $\psi_n$ are called penalizing functions and aim at modelling the elasticity and the viscous friction of the body’s surface. We tentatively distinguish in the following three different subclasses of such analysis among the available literature, but we agree that our classification is somewhat arbitrary and could be done differently (For instance chronologically). Our motivation to present these works is not only for the sake of completeness, but also to show to the non-mathematician reader how one may consider impact problems in another setting, and which problems arise from this different point of view. Moreover some of the cited works [476] [416] might be used in a control context since the considered external action is state dependent. In all the works that follow, it is assumed a constraint surface of codimension 1, i.e. $f(q) \in \mathbb{R}$.

2.2.1 The elastic rebound case

The one-dimensional problem

The problem considered is the following:

Problem 2.1 (Carriero-Pascali [93] [94]) A locally Lipschitz function $q(t)$ defined on $[0, T]$ is a solution of the one-dimensional rebound problem $P_f$ if:

a) $q \leq 0$ on $[0, T]$

b) $\langle \dot{q} - f, \varphi \rangle \geq 0$ for all $\varphi \in \mathcal{D}([0,T]), \varphi \geq 0, f \in L_1([0,T])$

c) If $q < 0$ then $\dot{q} - f = 0$ in the distributional sense.

d) $\forall t \in [0, T], \dot{q}(t^+) \text{ and } \dot{q}(t^-) \text{ exist, } \dot{q}(0^+) \text{ and } \dot{q}(0^-) \text{ exist, } \frac{1}{2}[\dot{q}(t^+)]^2 - \frac{1}{2}[\dot{q}(0^+)]^2 = \int_0^t f(\tau)\dot{q}(\tau)d\tau$, and the equality holds also for $\dot{q}(t^-)$.

The initial data are naturally assumed to be admissible, in particular if $q(0) = 0$ then $\dot{q}(0^+) \leq 0$, in order not to violate the constraint. a) is the unilateral constraint condition; b) means that the (impulsive) interaction force at impacts is a negative measure, i.e. there is a measure $\mu$ such that $\dot{q} - f = \mu$ and $\langle \mu, \varphi \rangle$ is either negative or zero if $q < 0$ as stated by c), which merely means that the smooth dynamical equations are verified (In example 1.1 we have $\mu = p_i \delta_{t_i}$ and $p_i = -2m\dot{q}(t^-) < 0$ when the shock is lossless. Adopting our notations we get $\dot{q} = \{\dot{q}\} + \sum \sigma_{q_k} \delta_{t_k}$, so that b) and c) becomes $\sigma_{q_k} < 0$ and $\{\dot{q}\} = f$. d) means that outside the constraint the system does not dissipate energy, and the impacts are lossless also ($c = 1$). In control theory language, the operator $f \rightarrow \dot{q}$ is passive, see appendix E. One therefore easily realizes that except for trivial cases ($f \geq 0$ and $q(0^-) = 0$ or
2.2. THE METHOD OF PENALIZING FUNCTIONS

$f \leq 0$) $\mu$ is atomic with atoms the zeros of $q$. No mention is made of this in these works, the first aim of which is to prove the existence of a solution to $\mathcal{P}_f$. Note however that implicitly the interaction force $\mu$ is an unknown of this problem, and this is a common feature of all these studies. Existence is studied in [93] by choosing in $\mathcal{P}_{n,f}$ a continuous penalizing function $\psi(q_n(t))$ such that $\psi_n(\zeta) = 0$ if $\zeta \leq 0$, $\psi(\zeta) > 0$ for $\zeta > 0$, $\psi_n \to +\infty$ on any compact interval of $(0, +\infty)$, and $\lim_{\zeta \to 0^+} \lim_{n \to +\infty} \frac{\psi_n(\zeta)}{\alpha_n(\zeta)} = +\infty$, with $\alpha_n(\zeta) = \int_0^\zeta \psi_n(\tau) d\tau$. It is easy to verify that the spring-like environment studied above fits within this framework (Thus clearly such penalizing functions aim at modelling a spring, but other examples with no clear physical meaning may be found, see [93]). Theorem 1 in [93] states that if $q_n$ is a solution to $\mathcal{P}_{n,f}$ that tends uniformly to a function $q$, then $q$ is a solution of $\mathcal{P}_f$, and that there exists at least one solution to $\mathcal{P}_f$ for each initial data, which is the uniform limit of some $\{q_n\}$.

The $n$-dimensional rebound problem

Basically Buttazzo and Percivale [84] [85] [427] [428] consider the same problem $\mathcal{P}$ as in problem 2.1. Similar results as in [93] [94] are obtained in [84] (a paper written between the other two) for the one-dimensional case. Condition c) is stated in [84] (as well as in [85] [427] [428] for the higher-dimensional case) as supp($\bar{q} - f$) $\subseteq \{t \in [0, T] : q(t) = 0\}$, i.e. the support of the distribution $\bar{q} - f$ is contained in the set of zeros of $q(t)$ when the particle attains the constraint. The works in [85] [427] [428] incorporate a variational formulation of the elastic bounce problem. We shall therefore come back in more details on these papers (see problem 3.1).

2.2.2 The general case

Let us describe now the work in [416], which considers a rebound problem with possible dissipative collisions. The problem is the following:

Problem 2.2 (Paoli-Schatzman [416]) Let $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous in $t$, $q$, $\dot{q}$ and Lipschitz in $q$, $\dot{q}$, uniformly with respect to $q$, $\Phi$ be a closed convex domain $\subset \mathbb{R}^n$, with nonempty interior and with smooth boundary $\partial \Phi$. Then $q : [0, T] \to \mathbb{R}^n$ is a solution of the problem $\mathcal{P}_F$: $\ddot{q} + \partial \psi_{\Phi}(q) \ni F(t, q, \dot{q})$, $\dot{q}_N(t_k^+) = -e_N \dot{q}_N(t_k^-)$ when $q(t_k) \in \partial \Phi$, where the subscript $N$ denotes the component normal to $\partial \Phi$, $e \in (0, 1)$, if $q$ fulfills the following:

a) $q$ is Lipschitz continuous and $\dot{q}$ is of bounded variation

b) $q \in \Phi$ for all $t \in [0, T]$

c) for any continuous function $v : [0, T] \to \Phi$, $\langle v - q, F - \dot{q} \rangle \leq 0$

d) the initial data verify $\dot{q}(0^+) = -e_N \dot{q}_N(0^-) + \dot{q}_N(0^-)$ when $\dot{q}(0^-)$ points outwards $\Phi$. 

The subscript \( T \) denotes the component tangent to \( \partial \Phi \). The definition of the subdifferential \( \partial \psi_T (q) \) is given in appendix D.

A solution \( q \) to \( P_F \) is shown to exist by studying the limit of the solutions of a sequence of approximating problems \( P_{n,F} \) with penalizing function \( \psi_n(q_n, \dot{q}_n) = \psi_{n,1}(q_n, \dot{q}_n) + \psi_{n,2}(q_n) \) using Yosida's approximants for the elastic term, and a discontinuous function for the viscous friction term (Such discontinuity is easily understandable looking at our example above: the total vector field of the system considering both contact and noncontact phases is continuous if only elastic terms are present, but it is not if viscous friction is added). It is worth noting that the viscous friction term contains a coefficient \( \varepsilon \) that is equal to \( \frac{1}{2 \sqrt{k}} \varepsilon \) in the preceding section on approximation when \( 0 < \varepsilon \leq 1 \), see (2.3). It is easy to show that \( P_{n,F} \) reduces to our example in the particular one-dimensional case, although the meaning of the approximants in higher dimensions is not obvious. Let us illustrate the theory developed in [416] on this simple one degree-of-freedom case, taken from [416].

**Example 2.1** In the case when \( n = 1 \), \( F \equiv 0 \) and \( \Phi = \mathbb{R}^+ \), the system can be written as

\[
\begin{aligned}
\ddot{q} + \partial \psi_{\mathbb{R}^+} (q) &\equiv 0 \\
\dot{q}(t_k^+) &= -\varepsilon(t_k^-) \\
\text{for all } t_k \text{ such that } q(t_k) &= 0
\end{aligned}
\]  

(2.5)

This is clearly the dynamical equations of a point striking an horizontal obstacle, with no external forces. The approximating problem chosen in [416] is

\[
\begin{aligned}
\ddot{q}_n + 2\varepsilon \sqrt{k_n} q_n \text{sgn}^{-}(q_n) + k_n q_n \text{sgn}^{-}(q_n) &= 0 \\
\text{where } k_n &\to +\infty \text{ as } n \to +\infty, \text{ sgn}^{-}(q_n) = \begin{cases} 0 & \text{if } q \geq 0 \\ 1 & \text{otherwise} \end{cases}, \text{ and } \varepsilon = -\frac{\ln(\varepsilon)}{\sqrt{\pi^2 + \ln^2(\varepsilon)}} \\
\text{(compare the value of the damping in this sequence of approximating problems with the value of the damping in (2.3)). } \\
\varepsilon \in (0, 1] \text{ is the restitution coefficient. Note that the function } \text{sgn}^{-}(q_n) \text{ allows to write contact and noncontact dynamics in a single equation. Note that the signs are reversed with respect to the examples we have treated above, since free motion occurs now for } q \geq 0. \text{ The initial conditions are chosen as } q_n(0) = a > 0 \text{ and } \dot{q}_n(0) = b < 0. \text{ Hence the mass point starts in the free-motion space with a velocity directed towards the obstacle. Denoting } \tau = -\frac{a}{b} \text{ and } \tau_n = \tau + \frac{\pi}{\sqrt{k_n(1-\varepsilon^2)}}, \text{ the solutions can be explicitly obtained and are (we recall them for convenience although they have been already obtained above)}
\end{aligned}
\]  

(2.6)

\[
q_n(t) = \begin{cases} 
 a + bt & \text{for } t \in [0, \tau] \\
 e^{-\varepsilon(t-\tau)\sqrt{k}} \sin \left[ (t-\tau)\sqrt{k(1-\varepsilon^2)} \right] \frac{b}{\sqrt{k(1-\varepsilon^2)}} & \text{for } t \in [\tau, \tau_n] \\
 -\exp \left( -\frac{\pi t}{\sqrt{1-\varepsilon^2}} \right) b(t-\tau_n) & \text{for } t \in [\tau_n, +\infty[ 
\end{cases}
\]  

(2.7)
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Then clearly \( q_n(t) \) in (2.7) converges towards

\[
q(t) = \begin{cases} 
  a + bt & \text{for } t \in [0, \tau] \\
  -eb(t - \tau) & \text{for } t \in [\tau, +\infty) \end{cases}
\]  

(2.8)

whose derivative possesses a discontinuity at \( t = \tau \).

In case when there is some external force acting on the particle in the example 2.1, then in general the equations are not integrable. But theorem 2 in [416] guarantees that the solutions of problem \( P_F \) possess a solution (not necessarily unique) whose first derivative is of bounded variation. The theorem is stated as follows

**Theorem 2.1 ([416])** Consider the system defined in problem 2.2. This system admits a solution in the sense defined as in problem 2.2, \( a, b, c, d \). This solution is obtained as the strong limit in \( W^{1,p}([0, T], R^n) \) for all \( p \in [1, +\infty) \), and weak * limit in \( W^{1,\infty}([0, T], R^n) \) \(^8\), when \( n \to +\infty \), of a subsequence of the sequence of solutions of

\[
\dot{q}_n + 2\varepsilon \sqrt{\kappa_n} G(q_n - P_\Phi(q_n), \dot{q}_n) + k_n(q_n - P_\Phi(q_n)) = f(t, q_n, \dot{q}_n)
\]

(2.9)

with \( q_n(0) = q_{n,0}, \dot{q}_n(0) = \dot{q}_{n,0}. \) \( P_\Phi \) denotes the projection on \( \Phi \), and \( G(v, w) = \begin{cases} (v^T w)v & \text{if } v \neq 0 \\
0 & \text{if } v = 0 \end{cases} \)

\( \nabla \nabla \)

Intuitively from the one degree-of-fredom case in example 2.1, the different terms of the approximating problems correspond to spring and damper-like actions. Roughly the proof proceeds in showing that \( P_{n,F} \) possesses \( L_\infty \)-bounded solutions that converge to \( q \), and that \( \psi_{n,1} \) and \( \psi_{n,2} \) converge weakly* towards measures \( \psi_1 \) and \( \psi_2 \) such that \( \dot{q} - f = \psi_1 + \psi_2 \), and c) is true. The last part of the proof is dedicated to study the rebound conditions (\(^9\)).

It is clear that since this study encompasses the case of the bouncing ball with \( 0 < e < 1 \), finite accumulation points in the impact sequence \( P_F \) are tolerated. This

\(^8\)\( W^{1,p}, 1 \leq p \leq \infty, \) denotes Sobolev spaces [71]. Any function \( f \in L_p \) possesses a distributional derivative that belongs to \( D^* \) (see definitions in appendices A and C). Then \( f \in W^{1,p} \) if this distributional or generalized- derivative coincides in \( D^* \) with a function in \( L_p \).

\(^9\)Concerning Sarkovskii's theorem (see [208] chapter 5) which is a powerful and relatively simple result on the dynamics of one-dimensional maps, Holmgren [208] writes *This interesting and beautiful result ... should give a pause to anyone who might think that all of the really good theorems which don't rely on the mathematical equivalent of a B-1 bomber were proven before the turn of the century.* Although we sincerely believe that Paoli and Schatzman's result is a major forward step concerning the mathematical foundations of dynamical systems with unilateral constraints, its proof really looks like a B-1 bomber: we therefore prefer to omit it. The interested readers are invited to read the original paper. An open and hard question however remains unanswered to: is it more difficult to do mathematics than running 1500 m. in 3 minutes and 40 seconds?.
is in contrast with the previous works in problems 2.1 and 3.1 that rely on energy preservation at impacts (see [93] theorems 3 and 4, [427] lemma 2.1, where it clearly appears that \( t_k < t_{k+1} \) for all \( k \) is a crucial property for uniqueness. Surprisingly enough, this "separation of impacts" assumption has also been currently done in the literature dealing with stability of the trajectories of measure differential equations, where accumulation points in \( \{t_k\} \) are also precluded in the available studies, as we shall see in chapter 7. Therefore they constitute a mathematical obstacle in the analysis of nonsmooth impact dynamics. However uniqueness is not proved in [416]. From a philosophical point of view, it seems strange that the addition of a simple dissipative term suffices to destroy the well-posedness of the dynamical problem. One is tempted to conjecture that uniqueness should hold in a certain sense to give such models a real physical meaning.

**Remark 2.5** Note that this way of formulating the problem implicitly supposes that the set \( f(q) > 0 \) is convex. From the developments in chapter 6, section 6.2, it is clear that Newton's rule should not be applied to the components along the Euclidean normal to \( \partial\Phi \) (i.e. simply \( \frac{\partial f}{\partial q} \)), but should take into account the inertia of the system: it is assumed in [416] that the tangential velocity remains continuous. As we shall see at the beginning of chapter 6, this kind of "complete" Newton restitution rule is not in general applicable. A special set of velocity coordinates must be used, defined from the kinetic energy metric, so that some generalized tangential velocities are indeed continuous, for frictionless constraints.

In fact in [416] the inertia matrix is supposed to be identity, so that the gradient on the configuration manifold is the one in \( \mathbb{R}^n \), and the configuration space is Euclidean. In particular this justifies the assumption that tangential velocities remain continuous at collisions. This seriously restricts the class of systems to be considered, but it is quite possible that the study in [416] extends to more general cases, as pointed out in [417]. However we can give another interpretation of this model. Indeed a Lagrangian system can be seen as a point in its configuration \( n \)-dimensional space, and the normal direction can be defined via the kinetic metric. Then one can think of a generalized \( n \)-dimensional particle, and \( \Phi = \{q : f(q) \geq 0\} \). Now when dealing with control problems, it is convenient on one hand not to view the system as a mechanical system evolving on a configuration manifold (in which case convexity is lost in general), but simply as a set of differential equations with solutions in the state space \( \mathbb{R}^n \). On the other hand, the domain defined by the unilateral constraint may not be convex, but change of coordinates can be applied so that it becomes convex in the new coordinates (see for instance remark 6.11). Thus the domain within which the \( n \)-dimensional particle evolves can be in general assumed to be convex without imposing too strong restrictions on the system. In summary, the most crucial problem for the application of Paoli-Schatzman's result in [416] is that of continuity of the "tangential" velocity. We shall see in chapter 8 that it is possible to design a feedback controller using a suitable coordinate transformation that allows us to recast the dynamics in the framework of theorem 2.1.
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Remark 2.6 Note that Φ may then be something quite different from the "physical" (or real-world) 2 or 3-dimensional Cartesian constraint. We shall see in chapter 6 important physical cases where ∂Φ is not smooth, but possesses singularities, whereas the real-world constraint seems at first sight quite regular.

Remark 2.7 This work might also be considered as a mathematical preliminary study for dynamical analysis of systems like particles bouncing inside a closed curve which are called in mathematical physics billiards [493] [47] [292]. Thus the problem is completely treated from the existence of solutions (but not uniqueness) to the trajectories global behaviour. It is note worthy that the case of nongradient ∂Φ is treated also in [417], when $T_L = 0$. The shock conditions are then stated simply from the energy conservation equation (see problems 2.1 and 3.1), which avoids the difficulty encountered with restitution rules at singularities, where the normal to ∂Φ does not reduce to a line in $\mathbb{R}^n$, but is the normal cone $N_\Phi$ (11). This proves the existence of a solution in the sense of problem 2.2, with $\dot{q} \in RCLBV$, for billiards with nongradient boundaries and elastic collisions (inside a polygon for instance, see [292] §6.4).

Remark 2.8 The result of Paoli and Schatzman extends the results in [292] (see chapter 1, theorems 1,2,3,4) which possess a local-in-time nature only (they are based on singular perturbation like analysis).

Briefly, the pioneering work in [476] treated the same problem, but with an energy conservation equality.

2.2.3 Uniqueness of solutions

We have seen that a mechanical system with unilateral constraints is basically the concatenation of flows and diffeomorphisms. We have also studied in chapter 1 some properties of the solution (continuity with respect to the initial data, autonomy property). Since it is known that for ODE's continuity with respect to initial data and uniqueness of the solutions are guaranteed by local Lipschitz continuity of the vector field12, it seems natural to try to extend these results to systems with unilateral constraints. In particular we have seen that continuity with respect to initial data properties are closely related to the occurrence of the discontinuities: if

---

10Note that closed is to be taken here in the physical or real-world meaning, whereas closed in the Paoli-Schatzman's problem is to be taken in the topological sense, i.e. the whole space itself is in fact closed.

11The sweeping process formulation described in section 5.3 treats the same problem, but with generalized dissipative shocks. The existential proofs are for the moment limited to codimension 1 constraints.

12Let us recall that both properties are not equivalent: for instance the solutions of $\dot{q} = q^3$, $q(0) = q_0$, $t_0 = 0$, are given by $q(t) = \pm \sqrt[3]{(-3/2) t + q_0^3}$. They are clearly continuous in $q_0$, but uniqueness fails.
the jumps are clearly separated \((t_{k+1} > t_k + \delta)\) then one is able to draw conclusions. Things are much less clear when finite accumulation points are possible. In the following we illustrate through an example given by Aldo Bressan in [69] how the external action on the system may create uniqueness problems. As the reader will see, such result is quite surprising because it is not in accordance with the properties of solutions of ODE's, which we are more confident with in general. Moreover it is not physically intuitive.

**Aldo Bressan’s counter-example**

We describe in this subsection the counter-example invented by Aldo Bressan [69] to prove that the addition of unilateral constraints can, even in very simple cases, yield nonuniqueness of solutions for some initial data. Let us consider the one-degree-of-freedom system

\[ \ddot{q} = Q(t), \quad q \geq 0, \quad \dot{q}(t_k^+) = -e \phi(t_k), \quad q(t_k) = 0 \tag{2.10} \]

and the function

\[ \phi(t) = \begin{cases} 
0 & \text{if } t \leq 0 \text{ or } t = \frac{1}{2^n} \\
\frac{1}{2^n} f \left[ 2^n f(t - \frac{1}{2^n}) \right] = \frac{1}{2^n} f(2^n t - 1) & \text{if } \frac{1}{2^n} < t < \frac{1}{2^n - 1} \\
g(t) & \text{if } t \geq 1
\end{cases} \tag{2.11} \]

where \(m \in \mathbb{N}, n \in \mathbb{N}\), and the functions \(f(\cdot)\) and \(g(\cdot)\) satisfy the following conditions

\[
\begin{align*}
&f : [0, 1] \to \mathbb{R}, \quad f(t) > 0, f(0) = f(1) = 0 \\
&\frac{df}{dt}(0) = -e^{2^{1-n} \frac{df}{dt}(1)}, \quad n \geq 3, \quad 0 < e \leq 1 \tag{2.12}
\end{align*}
\]

and

\[
\begin{align*}
&g(t) > 0 \text{ for } t \geq 1, \quad g(1) = 0, \quad \frac{dg}{dt}(1) = \frac{-e}{2^{n-1}} \frac{df}{dt}(1) > 0 \\
&\frac{d^k g}{dt^k}(1) = \frac{1}{2^{n-k}} \frac{d^k g}{dt^k}(1), \quad k = 2, 3, \ldots, n \tag{2.13}
\end{align*}
\]

Then the following is true

**Theorem 2.2** ([69]) The functions \(\phi(t), \phi'(t), \ldots, \phi^{(n)}(t)\) exist, are continuous and \(\phi(t) \geq 0\) for all \(t \in \mathbb{R}\), \(\phi(t) > 0\) for \(t > 0\) and \(t \neq \frac{1}{2^n}\). The first derivative \(\phi'(t)\) exists, is continuous for all \(t \neq \frac{1}{2^n}\) and

\[ \phi \left( \frac{1}{2^n} \right)^+ = -e \phi \left( \frac{1}{2^n} \right)^- \tag{2.14} \]
Hence one has defined a function $\varphi(\cdot)$ which is zero for negative times, then is composed on $[0, 1]$ of the concatenation of arches whose length tend to zero when $t$ tends to zero (with a sort of reversed accumulation point at $t = 0^+$). Now assume that $\dot{f}(t) < 0$ and that $\ddot{g}(t) \leq 0$. Then clearly $\ddot{\varphi}(t) \leq 0$ for all $t \in \mathbb{R}$. Roughly, the idea is to get $q(t) \equiv \varphi(t)$ (hence $Q(t) \equiv \varphi(t)$), so that the mass bounces against the constraint (see (2.14)) with a restitution coefficient $e$. This can be obtained as proved in the following

**Theorem 2.3 ([69])** Let us choose $Q(t) = \varphi(t)$ in (2.10). Then the functions $Q(t)$, $\dot{Q}(t)$, $\cdots$, $Q^{(n-3)}(t)$ are continuous. The trajectory $q(t) \equiv \varphi(t)$, $t \in \mathbb{R}$, possesses the initial conditions $q(0) = \dot{q}(0) = 0$ and satisfies $\dot{q}(t_m) = -eq(t_m)$, $m \in \mathbb{N}$, i.e. it is a trajectory of the dynamical system in (2.10). The trajectory $q(t) \equiv 0$ is also a solution of the dynamical equations in (2.10).

Such a result is quite surprizing, since the applied force is always negative. Hence if one initializes the system at rest on the surface $q = 0$, it should logically remain stuck on it. The underlying idea is to consider an external action $Q(t)$ which is negative, but such that its double integral $\varphi(t)$ is positive, is zero at $t = 0$, continuous, and with a first integral $\varphi(t)$ that jumps when $\varphi(t)$ attains zero ($t_m = \frac{1}{2^n}$). Clearly such a function is not obvious to construct, but theorem 2.2 guarantees its existence (an explicit construction of similar functions has been given for instance in [476] [93]). Let us however recall (see subsection 1.4.2) that in order to be complete, the dynamical equations in (2.10) must be completed by the condition

$$q(t) = \max[0, Q(t)] \text{ if } q(t_m) = 0$$

(2.15)

which allows us to recover the algebraic equation of motion when contact is permanently established, i.e. $Q(t) + F(t) = 0$, where $F(t)$ is the interaction force. When impacts occurs, this force has an impulsive form. When impacts cease, depending on the sign of the applied external force, it either becomes zero or equal to $-Q(t)$. Clearly if condition (2.15) is included, then the solution that starts at $q(0) = \dot{q}(0^-) = 0$ and with the above external force must be $q(t) \equiv \dot{q}(t) \equiv 0$ for all $t \geq 0$, since we get $\ddot{q}(t) \equiv 0$. We therefore tend to believe that such a nonuniqueness problem is rather pathological, and depends on the way the dynamics are formulated. In particular notice that the formulations of the dynamics in [93] [94] (see problem 2.1), in [476] [416] (see problem 2.2), and in [84] [85] [427] [428] (see problem 3.1) do not include a condition like in (2.15). Hence it is normal that the same sort of nonuniqueness problems appear with these formulations. Notice also that $e > 0$ in (2.12). This is not a surprize: we have already seen that the case of plastic collisions is quite different from elastic ones. In particular the sweeping process formulation that we shall describe in chapter 5 models generalized plastic impacts. Hence the existing counter-examples do not apply to this formulation.

For the sake of completeness of the exposition of the existing mathematical studies on systems with unilateral constraints, we nevertheless provide now the conditions that have been derived by some authors to prevent such nonuniqueness problems.
Chapter 2: Approximating Problems

Sufficient conditions for uniqueness

One-dimensional bounce Using a similar counter-example as [69] and [476], [93] show that uniqueness of solutions to $P_f$ may fail even for smooth $f$, and give some sufficient conditions for uniqueness to hold:

Theorem 2.4 ([93]) Let $f$ be absolutely continuous in $[0, T]$, $f \in L^1([0, T])$ and $f \geq 0$. Then if $f(0) > 0$ and $x(0) \neq 0$, $x(0^+) \neq 0$, there exists a unique solution to the dynamical problem formulated as in problem 2.1. If the initial conditions are admissible (i.e. if $x(0) = 0$ then $x(0^+) \leq 0$) and if $x^2(0^+) - 2x(0)f(0) > 0$, then the solution is unique also on $[0, T]$.

Notice that the signs are reversed here because the constraint is written as $x \leq 0$ in problem 2.1. The proof is based on several steps. The central fact is that there is a finite number of impact times $t_k$ on $[0, T]$. A sufficient condition for this is that $x^2(t_k^+)$ (or equivalently $x^2(t_k^-)$) be strictly positive. The conditions in theorem 2.4 aim at guaranteeing such condition, which can be verified using the conservation of energy equation.

In another work [94] the same authors prove the following result

Theorem 2.5 ([94]) The uniqueness of solutions to the Cauchy problem 2.1 is a generic property in $f \in L_1([0, T], \mathbb{R})$.

This result shows the prevalence of problems $P_f$, i.e. in fact of a particular type of second order differential equations, with unique solutions, as it is the case for ODE's with continuous right-hand-sides as Orlicz showed (see [554]). Note that when the impact is lossless the simple dynamical problem studied in chapter 1, example 1.1, is a particular case of problem $P_f$ in [94] with a zero external action: thus the results in [94] on uniqueness of the solution trivially hold. In [94] uniqueness is studied as follows: it is shown that for every solution $x(\cdot)$ to $P_f$ with force $f \in L_1$, simple there is a sequence $f_n \to f$ in $L_1$ so that $P_{f_n}$ has a unique solution $x_n \to x$ uniformly in $[0, T]$. Roughly one then uses density of the set of simple and $L_1$-bounded functions in the set of $L_1$-bounded functions to obtain the result in theorem 2.5.

$n$-dimensional bounce The first result on uniqueness has apparently been given by M. Schatzman in [476], who established the following result:

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13See [200] p.154: a property is generic in $E$ if the set $G$ of elements of $E$ which possess it, contains a dense (in $E$) open set. In a sense, one deduces from the property of density of a set in another one that there are elements of $G$ "almost everywhere" in $E$, although this "almost everywhere" has nothing to do with that relying on measure theory, see appendix B. Indeed there are dense sets of zero-Lebesgue measure, like the set of rational numbers in $\mathbb{R}$. In a sense, measure theory and topology don't seem to mesh properly.

14I.e. $f([0,T])$ is finite, i.e. consists of a finite set of numbers $c_1, \ldots, c_n$. In other words, the external action is piecewise constant, with a finite number of values.
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Theorem 2.6 ([476]) Consider the dynamical system in problem 3.1. Let \( V = \mathbb{R}^n \) with the Euclidean metric, \( f \in C^\infty(\mathbb{R}^n) \) and \( \frac{\partial f}{\partial q} = 0 \) on \( \mathbb{R}^n \). If the boundary of the set \( \Phi = \{ q : f(q) \geq 0 \} \) has strictly negative Gaussian curvature \(^{15}\), then the Cauchy problem 3.1 admits a unique solution on \([0, +\infty)\).

This result is extended in [427] [428]. Let us define the function (called the trace in [428]) \( T : [0, T] \times E \rightarrow \mathbb{R}^{3n+2} \) as (we drop the time argument in \( q \) and \( \dot{q} \) for simplicity)

\[
T(t, q) = \left( \frac{1}{2} \dot{q}^T M(q) \dot{q} + q \cdot \dot{q}, f(q) \dot{q}, 0 \right) \tag{2.16}
\]

where \( \dot{q}_r(t) = \frac{\partial f}{\partial q} M(q) \frac{\partial f}{\partial \dot{q}} \dot{q} - \frac{\partial f}{\partial q} M(q) \frac{\partial f}{\partial \dot{q}} \dot{q} \). \( E \) is the set of functions \( q(t) \) such that \( q(t) \) is Lipschitz continuous on \([0, T]\) and is a solution of the dynamical problem 3.1. The interest for defining such initial data for the Cauchy problem is that the usual initial data \( q(\tau_0), \dot{q}(\tau_0) \) are not stable when one considers the convergence of an approximating problem \( P_n \) towards the limit rigid problem \( P \). Indeed, if the function \( q_n \) is a solution of \( P_n \), and if the sequence \( \{q_n\} \) converges uniformly towards \( q(t) \) which is a solution of \( P \), then it is not guaranteed that the initial data of \( P_n \) (i.e. \( q_n(\tau_0), \dot{q}_n(\tau_0) \)) converge uniformly towards \( q(\tau_0), \dot{q}(\tau_0) \). This is due to possible discontinuities in the velocity. On the contrary, the initial trace in (2.16) allows one to avoid this difficulty. The trace for \( P_n \) is defined as

\[
T_n(t, q_n) = \left( \frac{1}{2} \dot{q}_n^T M(q_n) \dot{q}_n + \alpha_n(f(q_n)), q_n, \dot{q}_n, f(q_n) \dot{q}_n, \frac{1}{n} \dot{q}_n \right) \tag{2.17}
\]

The function \( \alpha_n(\cdot) \) is defined after problem 2.1 (see also problem 3.1 in chapter 3) and may represent a spring-like action (but this is not necessary, other functions with no physical meaning can be used).

The traces \( T \) and \( T_n \) are continuous with respect to \( t \) for every \( q \in E \) and \( q_n \in E_n \) respectively (\( E_n \) is the set of functions \( q_n(t) \) such that \( q_n(t) \) is Lipschitz continuous on \([0, T]\) and is a solution of the dynamical problem \( P_n \)). When \( f(q) > 0 \), i.e. when the dynamics are smooth, assigning \( T(\tau_0, q(\tau_0)) \) is equivalent to assigning the Cauchy data \( q(\tau_0), \dot{q}(\tau_0) \) \([85]\).

Now fix some \( t_0 \in [0, T] \) and some element \( b \) of \( B \triangleq T(\{t_0\} \times E) \) (i.e. the image of the couple \( \{t_0\}, E \)) under the mapping \( T \), or, in other words, take any \( q(t) \in E \) estimated at \( t_0 \). Define now the set

\[
G(t_0, b) = \{ q \in E : T(t_0, q) = b \} \tag{2.18}
\]

This is therefore the set of all functions \( q(t) \) which solve the dynamical problem \( P \), and whose initial trace is equal to \( b \). Then the following is true

\(^{15}\)Let a surface \( S \) in \( \mathbb{R}^3 \) be given by \( q_3 = f(q_1, q_2) \), with \( \frac{\partial f}{\partial q_1}(q_{10}, q_{20}) = \frac{\partial f}{\partial q_2}(q_{10}, q_{20}) = 0 \) (these two vectors span the tangent plane to \( S \) at \( P \)) and the \( q_3 \)-axis is normal to \( S \) at \( P = (q_{10}, q_{20}, q_{30}) \). Then the Gauss or total curvature of \( S \) at \( P \) is equal to the determinant of the Hessian of \( f(q_1, q_2) \) at \( P \), i.e. the matrix \( \frac{\partial^2 f}{\partial q_1 \partial q_2} \in \mathbb{R}^{2 \times 2} \). It is for instance easy to verify that a plane given by \( q_3 = a_1 q_1 + b q_2 \) has zero total curvature at any of its points. The ideas generalize for higher dimensions.
Chapter 2. Approximating Problems

Theorem 2.7 [428] If $f : \mathbb{R}^n \to \mathbb{R}$ is real analytic\(^{16}\), then for each $t_0 \in [0, T]$ and for each $b \in \mathcal{B}$, the set $G(t_0, b)$ is a singleton.

In other words, given a set $\mathcal{T}(t_0, q(t_0))$ of initial data, the set of solutions of the Cauchy problem is reduced to one point (one function).

---

\(^{16}\)Recall that smoothness is not sufficient for a function to be analytic, i.e. to be equal to its Taylor expansion at any point.
Chapter 3

Variational principles

In most of the textbooks on analytical mechanics containing a chapter on impacts, e.g. [529] [141] [278] [424] [161] [38] [273] [376] [429], it is shown that one can derive extensions to basic tools of classical mechanics, e.g. the fundamental principle of dynamics and the Lagrange equations in the case of impacts phenomenons. From the rigid body assumption, it can be shown that the theorems of the linear and angular momentum can be easily retrieved (see chapter 4), and that they do not depend on the chosen frame which may be Galilean or not \(^1\) [529] [376], provided the velocity field associated to this frame is continuous [376] [429].

Since most of the modern robot control literature (rigid or flexible manipulators as well as bipede robots) is based on the Lagrangian formulation of the dynamics, it seems fundamental to investigate what happens with systems subject to unilateral constraints. Thus we focus our attention in section 3.2 on the variational formulation of impact dynamics via Hamilton’s principle. Before this, we spend a certain time with another minimization principle of mechanics, Gauss’ least action principle and its extension to systems with unilateral constraints, which has received some attention in the literature for a long time [51] [52], and until recently [375] [373] [320] [459] [497].

3.1 Gauss’ principle

Gauss’ least action principle is one of the several variational principles of mechanics, within which one can interpret the dynamics of classical mechanical systems (by classical we mean mainly here systems with smooth dynamics) [302]. Let us first recall its formulation for systems subject to holonomic (bilateral) constraints. In fact, Gauss’s principle is a reinterpretation of d’Alembert’s principle\(^2\) into a mini-

\(^1\)This may constitute another advantage of algebraic methods based on rigid bodies assumption compared to differential methods.

\(^2\)which is an extension of the principle of virtual work, according to which a system with holonomic bilateral constraints is in equilibrium if and only if the total virtual work of the interaction
minimum principle. For a system of \( n \) particles with coordinates \( q_i \in \mathbb{R}^3 \) and masses \( m_i \), submitted to external forces \( F_i \), Gauss’ principle states that the function

\[
G(\ddot{q}) = \sum_{i=1}^{n} \frac{1}{2m_i} (F_i - m_i \ddot{q}_i)^2
\] (3.1)

is minimum for the actual motion. This recasts the research of acceleration at each instant of time into a quadratic programming problem. If the system is free of any constraints, then trivially one retrieves Newton’s law of motion, i.e. \( m_i \ddot{q}_i = F_i \).

In the case of a general \( n \)-degree-of-freedom Lagrangian system, the function \( G(\ddot{q}) \) takes the form

\[
G(\ddot{q}) = \frac{1}{2} \dot{q}^T M(q) \ddot{q} - \dot{q}^T (-C(q, \dot{q}) \ddot{q} - g(q) + Q)
\] (3.2)

where the different terms are defined in example 1.3, equation (1.10). It turns out that Gauss’ principle is also valid for systems subject to unilateral constraints. This generalization has been considered in [375][373][320][51][52][497].

Let us consider a \( n \)-degree-of-freedom system subject to \( f_i(q) \geq 0, \ i \in \{1, \cdots, m\} \). Then, as we shall see in chapter 5, section 5.4, the Lagrange multipliers \( \lambda_i \) and the system’s state must verify the set of conditions

\[
\lambda_i \geq 0 \quad \ddot{q}_i \geq 0 \quad \lambda^T \ddot{q} = 0
\] (3.3)

where it is assumed that at the instant when these conditions are stated, \( f(q) = 0 \). Using the systems dynamical equations, one can easily introduce the acceleration \( \ddot{q} \) into conditions (3.3). It can be proved (see e.g. section 5.4, theorem 5.2) that these conditions define a unique set of values for the acceleration \( \ddot{q} \), considered as the unknowns of the problem at the given instant of time when the conditions are considered. Moreau [373][375] proved that this solution \( \ddot{q} \) is also the one that minimizes the function \( G \) in (3.2). Hence Gauss’ principle is valid for systems with unilateral frictionless constraints.

A more abstract point of view of Gauss’ principle is also given in [375], using tools from convex analysis. This is based on the fact that the inequalities \( \ddot{q}_i(q) = \nabla_q f_i(q)^T \ddot{q} + \frac{d}{dt} \left( \nabla_q f_i(q)^T \right) \dot{q} \geq 0 \) in (3.3) can be rewritten as

\[
\langle \nabla_q f(q), x \rangle + \frac{d}{dt} \left( \nabla_q f_i(q)^T \right) \dot{q} \geq 0
\] (3.4)

where the scalar product is defined as \( \langle x, y \rangle = x^T M^{-1}(q) y \), and \( x \) in (3.4) is equal to \( M(q) \ddot{q} \). The inequalities in (3.4) define a closed convex polyhedral region \( C \) in a \( n \)-dimensional Euclidean space with the metric defined from \( \langle \cdot, \cdot \rangle \). Then Gauss’ principle states that the real motion is such that \( x \) in (3.4) is, in \( C \), the nearest point of forces vanishes, where the virtual displacements are arbitrary but consistent with the constraints. D’Alembert’s principle adds the forces of inertia to the interaction forces in the principle of virtual work [302].
3.2. LAGRANGE’S EQUATIONS

from the known point \( z = -C(q, \dot{q})\dot{q} - g(q) + Q \), i.e. \( x = \text{proj}_C z \) (see definition D.5).

If \( \Psi(x) \) denotes the indicatrix function of the set \( C \) (see appendix D, definition D.1), then this may be written as

\[
x = \text{prox}_{\Psi} z
\]

Furthermore, by defining the dual function \( \Xi \) of \( \Psi(x) \) as \( \Xi(y) = \sup_{x \in C} \langle x, y \rangle = \sup_{x \in C} \langle x, y \rangle \) (see definition D.6), one finds that the Lagrange multipliers \( \lambda \) satisfy

\[
-\nabla_q f(q) \lambda = \text{prox}_{\Xi z}
\]

In other words, \( -P_q = -\lambda \nabla_q f(q) \) is the proximal point \( \text{prox}_{\Xi z} \), i.e. the nearest point in \( \Xi(y) \) from the known point \( z \).

3.1.1 Additional comments and studies

Sinitsyn [497] studies a system composed of \( n \) mass-points. He shows that the real acceleration of the system \( \ddot{q}_i \), satisfies \( \sum_{i=1}^n m_i (\ddot{q}_i - \ddot{q}_0)^2 \leq \sum_{i=1}^n m_i (\ddot{q}_i - \ddot{q}_0)^2 \), where \( \ddot{q}_0 \) is the acceleration the system would have with no constraint, and the virtual acceleration \( \ddot{q}_i \) corresponds to virtual displacements \( \delta q \) consistent with the constraints (i.e. \( \nabla_q f_j(q) \delta q < 0 \)).

Other variational principles, like Maupertuis', can be derived for systems with unilateral constraints. This is investigated in [292] for elastic impacts. Papastavridis [420] shows that the impulsive form of some well-known finite motion equations, like Routh-Voss', Maggi's, Hadamard's, Boltzmann-Hamel's, Chaplygin-Voronets' and Appell's equations can be derived. Bahar [26] studies the extension of the differential variational principle of Jourdain (JVP) to rigid body shock dynamics, making use of quasi-velocities. He also points out the connection between the JVP and Gauss' principle, and the possible applications of such theoretical results: reconstruction of pre-impact velocities from post-impact data (which is very important in studying for instance vehicle accidents, see also Brach [54] for such application). Kirgetov [279] [280] [281] studies dynamics of systems of \( n \) particles subject to frictionless unilateral constraints. The collisions are assumed to be elastic. It is shown [280] that among all the states consistent with the constraints \( f(q) \geq 0 \), and satisfying \( \nabla_q f(q)^T \dot{q}(t_k^+) = -\nabla_q f(q)^T \dot{q}(t_k^-) \), the real state \( (q, \dot{q}) \) is the one that minimizes the function \( \sum_{i=1}^n \frac{m_i}{2} \left( \ddot{q}_i(t_k^+) - \ddot{q}_i(t_k^-) \right)^2 \).

3.2 Lagrange’s equations

From a general point of view, a variational problem in mechanics can be formulated as follows [608]: given a Lagrangian \( L \), a suitable class of admissible curves \( q \) with given endpoints, find the minimum of some quantity called the action integral \( I(q) = \int_{t_0}^{t_1} L(t, q(t), \dot{q}(t)) dt \) (3). In this section, we first treat this problem as if there

\footnote{This kind of variational problems is often referred to as the Lagrange problem [438].}
were no difficulties in applying variational techniques to systems subject to impacts, i.e. we simply take the Lagrange equations and suppose that a generalized external impulsive force acts on the system (We have already noted a significant difference between this problem and the problem with unilateral constraints in chapter 1). Then we discuss about the variational approach to systems with unilateral constraints.

### 3.3 External impulsive forces

The Lagrange equations of the system submitted to a generalized force $F + p\delta t_k$, where $F$ represents all the generalized forces without taking into account the impulsive ones, are given by

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = F + p\delta t_k.$$ 

Thus using the notations introduced in chapter 1 we get:

$$\frac{\partial^2 T}{\partial q^2} \left( \{\dot{q}\} + \sigma_q \delta t_k + \sigma_q \delta t_k \right) + \frac{\partial^2 T}{\partial q \partial \dot{q}} \left( \{\dot{q}\} + \sigma_q \delta t_k \right) - \frac{\partial (T - U)}{\partial q} = F + p\delta t_k \quad (3.7)$$

from which we deduce:

$$\frac{\partial^2 T}{\partial q^2} \sigma_q = 0 \quad \text{and} \quad \frac{\partial^2 T}{\partial q \partial \dot{q}} \sigma_q + \frac{\partial^2 T}{\partial \dot{q}^2} \sigma_q = p \quad (3.8)$$

In case the kinetic energy is a quadratic form of the velocity, $\frac{\partial^2 T}{\partial q^2}$ is the positive definite inertia matrix and we obtain:

$$\sigma_q = 0 \quad \text{and} \quad \frac{\partial^2 T}{\partial q \partial \dot{q}} \sigma_q = \frac{\partial T}{\partial q} (t^+_k) - \frac{\partial T}{\partial q} (t^-_k) \quad (3.9)$$

so that finally the Lagrange equations become in case of external impact:

$$\frac{\partial^2 T}{\partial q^2} \{\dot{q}\} + \frac{\partial^2 T}{\partial q \partial \dot{q}} \dot{q} - \frac{\partial (T - U)}{\partial q} = F \quad (3.10)$$

$$\frac{\partial T}{\partial q} (t^+_k) - \frac{\partial T}{\partial q} (t^-_k) = p$$

Note that (1.11) is a particular case of (3.10). In general (3.10) is obtained by integrating the Lagrange equations over infinitely short time intervals of shocks durations \[379\], \[612\], \[14\], \[13\], \[420\]. Note that (3.10) can be extended to the case when the impulsive forces are any distribution in $D^*$. 

#### 3.3.1 Example: flexible joint manipulators

Let us illustrate equations (3.10) with the case of elastic joint manipulators. These systems have been deeply studied in the robots control literature since they represent a nice example of Lagrangian systems with less inputs than degrees of freedom (The inputs are the torques at the joints). For a survey see \[510\] and \[76\]. Two dynamical

---

4The term $\delta t_k \delta t_k$ (see (1.11)) does not appear explicitly in (3.7) but is contained in the second term of the left-hand-side.
3.3. EXTERNAL IMPULSIVE FORCES

models have been used to design stabilizing controllers for such systems. The first model has been obtained in [511]:

\[ M(q_1)\ddot{q}_1 + C(q_1, \dot{q}_1)\dot{q}_1 + g(q_1) = K(q_2 - q_1) \]
\[ J_m\ddot{q}_2 + K(q_2 - q_1) = u \]

(3.11)

where \( q_1 \in \mathbb{R}^n \) are the links angles, \( q_2 \in \mathbb{R}^n \) are the motorshafts angles, \( K \in \mathbb{R}^{n \times n} \) is the joint stiffness matrix, constant and diagonal, \( J_m \) is the motors inertia matrix, \( u \) is the control input. Note at once that the control problem for such systems has been more challenging than for rigid manipulators. Mainly this is due to the fact that the coordinates to be stabilized appear in the first equation in (3.11), and the control appears in the second equation.

Let us now assume that the system is submitted to a unilateral constraint \( f(q_1) \geq 0 \), with \( f(q_1) \in \mathbb{R} \) and smooth. Hence \( \nabla_q f(q_1) = \begin{pmatrix} \frac{\partial f}{\partial q_1} \\ 0 \end{pmatrix} \), and the dynamical equations at the impact times are given by

\[ M(q_1)\ddot{q}_1(t_k) = \frac{\partial f}{\partial q_1}p_k \]
\[ J_m\ddot{q}_2(t_k) = 0 \]

(3.12)

where \( p_k \in \mathbb{R} \) is the impulsive Lagrange multiplier corresponding to the normal interaction impulse at the impact (we assume a frictionless surface \( f(q_1) = 0 \)). One concludes that \( \dot{q}_2 \) remains continuous at the impact.

The model in (3.11) is in fact obtained by neglecting the effects of the link velocities \( \dot{q}_1 \) in the kinetic energy of the motorshafts. When these effects are taken into account, then one obtains a more complex model [539], given by

\[ M_{11}(q_1)\ddot{q}_1 + M_{21}(q_1)\dot{q}_2 + C_{11}(q_1, \dot{q}_1)\dot{q}_1 + C_{12}(q_1, \dot{q}_1)\dot{q}_2 + g(q_1) = K(q_2 - q_1) \]
\[ M_{21}(q_1)\dot{q}_1 + M_{22}\dot{q}_2 + C_{21}(q_1, \dot{q}_1)\dot{q}_1 + K(q_2 - q_1) = u \]

(3.13)

where \( M_{22} = J_m \). The difference between the two models in (3.11) and (3.13) is that in (3.13) the inertia matrix is no longer diagonal. Hence there are acceleration cross terms in the dynamical equations. Note that from a control point of view, this drastically complicates the problem. For instance the model in (3.11) is static state feedback linearizable, whereas the one in (3.13) is dynamic state feedback linearizable only (see [399] for definitions). Now the impact dynamical equations for (3.13) are given by

\[ M_{11}(q_1)\ddot{q}_1(t_k) + M_{21}(q_1)\dot{q}_2(t_k) = \frac{\partial f}{\partial q_1}p_k \]
\[ M_{21}(q_1)\dot{q}_1(t_k) + M_{22}\dot{q}_2(t_k) = 0 \]

(3.14)

It is obvious from (3.14) that \( \dot{q}_2 \) may be this time discontinuous at the impact, contrarily to the previous case where we deduced from (3.12) that it had to remain...
continuous. This is rather surprising if we think a little of the mechanical structure of such systems: the impact occurs between the last link (the end-effector) and the environment. By which dynamical effect could the motorshafts (which are in some sense "protected" by the elasticity) possess a discontinuous velocity? Now let us go a little deeper into the structure of the matrix $M(q_1)$. It can be shown [539] that this matrix has the following form

\[
\begin{pmatrix}
0 & M_{21,12} & M_{21,13} & \ldots & M_{21,1n} \\
0 & 0 & M_{21,23} & \ldots & M_{21,2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & M_{21,n-1,n} \\
0 & 0 & 0 & \ldots & 0
\end{pmatrix}
\]

(3.15)

Now take the transpose of the matrix in (3.15) and introduce it in the second dynamical equation in (3.14). From the fact that $M_{22}$ is diagonal, it follows that $\sigma_{q_2,1}(t_k) = 0$. But the rest of the components of $q_2$ may possess a discontinuity at the impact time. Indeed we get $M_{21,12}\sigma_{q_1,1}(t_k) + J_{22}\sigma_{q_2,2}(t_k) = 0$, $M_{21,13}\sigma_{q_1,1}(t_k) + M_{21,23}\sigma_{q_1,2}(t_k) + M_{22,23}\sigma_{q_2,3}(t_k) = 0$ and so on. The only component of $\sigma_{q_i}$ which does not influence $q_2$ is $q_{1,i}$. As we shall see in chapter 4, some collision rules will have to be applied to $q_1$. Hence in general some components of $q_1$ will be discontinuous. The jumps in $q_2$ have then to be computed from the second equation in (3.14), and the percussion is deduced from the first equation in (3.14). We refer the reader to remark 6.2 for further investigations.

In conclusion, the simplified model in (3.11) always yields continuous motorshaft velocities, whereas the complete model in (3.13) may yield discontinuous motorshaft velocities. This is due to dynamical coupling that exists in the robot with elastic joints. This may have some practical consequences, since it means that joint compliance does not really prevents the impulsive action from acting on the actuators angles (This of course does not mean at all that the actuators suffer from an impulsive action, see the second equation in (3.14)). Note that in certain cases, like parallel drive manipulators [607], where all the actuators are mounted at the fixed base of the robot, then the simplified model is the exact one. In this sense it can be stated that parallel drive manipulators are less sensitive to impacts than other robots.

3.4 Hamilton's principle and unilateral constraints

3.4.1 Introduction

We now turn our attention on a much more difficult problem, i.e. the variational formulation of impact dynamics. This pertains to the field of calculus of variations when the data (the functional to be minimized and/or the space of admissible
3.4. **HAMILTON’S PRINCIPLE AND UNILATERAL CONSTRAINTS**

Curves) suffer from drastic perturbations (discontinuities, impulses...) [118] [389] [608]. Classically, two problems arise in such minimization problems [608]: i) Nature of the curves (i.e. in which space have the admissible curves \( q \) to be defined?) to insure existence of an extremum (minimum) of the integral action \( \mathcal{I} \) ii) Necessary and sufficient conditions to be verified by the extremals (Lagrange equations, transversality and Erdmann–Weierstrass corner conditions).

Roughly, point i requires that the nature of the admissible curves be specified as well as the topology associated to the space to which they belong, since existence result can be proved via compactness of the level sets and lower-semicontinuity of the functional to be minimized, which is known as Tonelli’s direct method [118]. The direct method of variational calculus is based on the fact that a function \( F(\cdot) \) defined on a compact set \( C \) and lower semicontinuous (\( \delta \)) on \( C \), attains its lower bound on \( C \), i.e. there exists \( x_0 \in C \) such that \( F(x_0) = \inf_{x \in C} F(x) \). This also applies to functionals \( I(q) \), and is known as Tonelli’s direct method which allows to prove the existence of a minimizing curve \( q \):

**Theorem 3.1 (Tonelli [118])** Assume that \( I : E \rightarrow \mathbb{R} \) is coercive and lower semicontinuous. Then \( I(\cdot) \) has a minimum point in \( E \).

By point is meant here an element of \( E \), i.e. a curve if \( I \) is a functional and \( E \) a space of functions. In case \( E \) is a metric space, then \( I \) is coercive if its level sets , i.e. the sets \( \{ q \in E : I(q) \leq \alpha \}, \alpha \in \mathbb{R} \), have a compact closure [118] definition 1.12 (\( \delta \)). Also a bilinear form \( a(u, y) : L^2 \times L^2 \rightarrow \mathbb{R} \) is coercive if \( a(u, u) \geq \alpha ||u||_2^2 \) for some \( \alpha > 0 \) [71]. Of course the great difficulty in proving such existential results lies in the choice of suitable spaces of admissible curves (i.e. \( E \)), together with a suitable topology (i.e. a notion of convergence) that allows to prove coercivity and lower semicontinuity of \( I \) in \( E \). It is another problem to derive necessary and/or sufficient conditions that an extremal point must satisfy: this is point ii).

As we already saw in the preceding chapters, a natural set for such variational problems is a subset of

\[
Q = \{ q(\cdot) : q \in C([t_0, t_1], \mathbb{R}^n), \dot{q} \in RC BV([t_0, t_1], \mathbb{R}^n), \ddot{q} \in D^* \}
\]

(3.16)

endowed with a suitable topology for compactness (in view of the existential results described in chapter 2, we can even impose \( q(\cdot) \) Lipschitz continuous). Note that it is also customary in variational calculus to use as basic spaces of admissible curves Sobolev spaces [71] [118], i.e. \( q \in W^{1,p}([t_0, t_1]) \), see appendix C for the definition of such spaces of functions. In some other more general problems [389] that could fit with ours if \( q \) was allowed to be discontinuous (thus \( L \) would contain singular

---

\(^5\) "Every problem of the calculus of variations has a solution, provided the word "solution" is suitably understood." (D. Hilbert), so that ... in variational problems the original setting must be modified in accordance with the needs of an existence theory. [608] p.218.

\(^6\) See definition D.7 in appendix D.

\(^7\) Recall that if the level sets are compact, the function is said to be proper [504] definition 4.6.1. Hence properness implies coerciveness.
distributions like in Bressan’s hyper-impulsive systems), the basic space is that of equivalent classes of $RCLBV$ functions.

In order to get more insight on this point, consider the following one-degree-of-freedom simple example, where the dynamics can be integrated at hand: let a disc with radius $\frac{d}{2}$ move without friction on a horizontal plane between two parallel rigid "walls", situated at a distance $d + 2\varepsilon$ one from each other. This may represent the dynamics of a mechanism with clearance, or with a "bi-unilateral" constraint of the form $c_1 \leq f(x) \leq c_2$, considered for instance in [438] chapter 3, or a simplified Fermi accelerator model [210]. There is conservation of energy, and the initial conditions on position and velocity are such that the problem is well-posed. Then the graph of the position of the disc center with respect to time is a "saw-toothed" or "zig-zag" diagram. The singular points correspond to the impact times, with $t_k = \frac{(2k+1)\varepsilon}{2\alpha}$ (assuming $x_0 = 0$). By letting $\varepsilon$ approach zero, this curve tends towards an infinitesimal zig-zag curve, that is not a curve, but a generalized curve [608] chapter 6, i.e. an element of the dual space of continuous functions (Young’s generalized curves [608]) or smooth functions (Schwartz’s distributions). It is worth noting that in this limit case of zig-zag curve with $\varepsilon = 1$, the corresponding velocity $\dot{x}$ is not of bounded variation on any interval of strictly positive measure as $\varepsilon \to 0$, since the percussion magnitude is $2m|x_0| > 0$ at each impact and the flight-time is $\Delta_k \overset{\text{def}}{=} \frac{2}{n|x_0|}$. Hence by letting $n \to +\infty$, the total variation of the velocity on a bounded interval $I$ grows unbounded. Since anyway the zig-zag curve is measurable, it defines a Schwartz’ distribution and thus possesses infinitely many distributional (or generalized) derivatives. This may not be of interest for us, since we are rather interested (mainly for stability notions and control purposes) by solutions which are $RCLBV$. It seems that this "pathological" saw-toothed case has not been revealed elsewhere in the literature on impact dynamics. One may be tempted to conjecture that problems with unilateral constraints $f(q) \leq 0$ always possess a solution in $Q$, provided some mild conditions are imposed on the constraint $f(\cdot)$. Recall that it is difficult in general to assert that the solutions are in $Q$: existential results need deep mathematical investigations, see chapter 2, and section 5.3. For instance, it is true that $\forall I \subset R$, $I$ compact, then $\int_I T(t)dt < +\infty$, i.e. the kinetic energy is locally integrable. However this implies that $\dot{q} \in L_2[I]$, which in turn does not imply that $\dot{q}$ is of bounded variation. One could think of the solution as being a measurable function, but then we lose the interpretation of the interaction impulses as being countable. As we shall see velocities of bounded variation lend themselves very well to some sort of stability results, and are from this point of view quite convenient to work with.

---

The saw-toothed functions $x_n(t)$ converge uniformly towards the function $x \equiv 0$ [154] p.64: indeed $\sup_{t \in I} |x_n(t)| = \frac{1}{n} \to 0$ when $n \to +\infty$. However the sequence $\{x_n\}$ does not converge towards $\dot{x} \equiv 0$, not even pointwisely since $|x_n(t)| = 1$ for almost all $t$. Note that this is reassuring: if $x_n \to 0$ then $\dot{x}_n \to 0$ in the distributional sense so that no impacts occur in the limit. Another point of view is that the infinitesimal zig-zag curve can be described by assigning the pair of slope $+1$ and $-1$ at each point with a probability $\frac{1}{2}$ [608] p.160. This makes it clear that the saw-toothed function does not converge to the function $\dot{x} \equiv \dot{x} \equiv \ddot{x} \equiv \ldots \equiv 0$ but to something else in a space of "generalized curves".

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\footnote{Let us define $n = \frac{1}{\varepsilon}$. The saw-toothed functions $x_n(t)$ converge uniformly towards the function $x \equiv 0$ [154] p.64: indeed $\sup_{t \in I} |x_n(t)| = \frac{1}{n} \to 0$ when $n \to +\infty$. However the sequence $\{x_n\}$ does not converge towards $\dot{x} \equiv 0$, not even pointwisely since $|x_n(t)| = 1$ for almost all $t$. Note that this is reassuring: if $x_n \to 0$ then $\dot{x}_n \to 0$ in the distributional sense so that no impacts occur in the limit. Another point of view is that the infinitesimal zig-zag curve can be described by assigning the pair of slope $+1$ and $-1$ at each point with a probability $\frac{1}{2}$ [608] p.160. This makes it clear that the saw-toothed function does not converge to the function $\dot{x} \equiv \dot{x} \equiv \ddot{x} \equiv \ldots \equiv 0$ but to something else in a space of "generalized curves".}
3.4. HAMILTON'S PRINCIPLE AND UNILATERAL CONSTRAINTS

Note that first of all the problem has to be well formulated: this is what we discuss now. Let us illustrate how corners\textsuperscript{9} may "naturally" be contained in a variational problem. For instance, assume the integrand of $I(x)$ is equal to $L(x, x) = (1 + x^2)(1 + \left[\dot{x}^2 - 1\right]^2)$ with endpoint conditions $0, 1, x(0) = x(1) = 0$. Then the minimizing curve is "naturally" an infinitesimal zig-zag \cite{608} p.159. We may say that the integral action "contains" the irregularities. This is not the case for a mechanical problem if a classical Lagrangian is considered as the \textit{Erdmann-Weierstrass corner conditions} \cite{438} show. These conditions yield

$$\frac{\partial L}{\partial \dot{q}}(t_k^+) = \frac{\partial L}{\partial \dot{q}}(t_k^-)$$

(3.17)

and

$$\left(L - q^T \frac{\partial L}{\partial \dot{q}}\right)(t_k^+) = \left(L - \dot{q}^T \frac{\partial L}{\partial \dot{q}}\right)(t_k^-)$$

(3.18)

Condition (3.17) yields $M(q)\dot{q}(t_k^+) = M(q)\dot{q}(t_k^-)$, while condition (3.18) yields $T(t_k^+) = T(t_k^-)$, where $T = \frac{1}{2}q^T M(q)q$. From the first condition, the generalized acceleration $M(q)\sigma_q(t_k) = 0$ so that the corresponding percussion is zero also, see example 1.3. Hence $\dot{q}$ is continuous. Further, consider now the bouncing ball (on a fixed table) example, and assume that one wants to search for the extremals of a classical variational problem $I(q) = \int_{t_0}^{t_1} L(q, \dot{q})dt$, with fixed endpoints and the unilateral condition $q \geq 0$: then for any $t_0, t_1, q(t_0) > 0, q(t_1) \geq 0$, the minimization process will always lead to a smooth solution curve, because the endpoint conditions uniquely determine the initial data; roughly speaking, the solution will always be such that it takes the whole interval $[t_0, t_1]$ to reach the constraint $q = 0$. As a consequence the impacts will always be absent of such a formulation! It is therefore necessary for the solution curve to contain impacts that the endpoint conditions be modified by fixing for instance $t_0, t_1, q(t_0)$ and $\dot{q}(t_0)$, a choice that is also made in \cite{85}, see problem 3.1 below\textsuperscript{11}. Notice that this \textit{a priori} fixes $q(t_1)$ if we assume uniqueness of the solutions. Then the constraint $q = 0$ will in general be reached at $t_2 < t_1$ and with a nonzero velocity $\dot{q}(t_2)$. Still this is not sufficient as the corner conditions indicate. The problem thus must be transformed to one of the form in a or b below.

Let us now formulate the variational problem. Firstly notice that in order for this problem to make sense, either the integral action to be minimized must "contain" the impacts, or the space of admissible curves $Q$ has to be modified. Basically two paths may be followed:

\textbf{a}) Search for $m_Q(I) = \min I(q), \quad q \in \hat{Q} \triangleq \{q \in Q, f(q) \geq 0, \dot{q}_N(t_k^+) = -e\dot{q}_N(t_k^-) \text{ with } f(q(t_k)) = 0, \text{admissible initial conditions}\}$

\textsuperscript{9}i.e. nondifferentiable points.

\textsuperscript{10}Obviously if the endpoint conditions do not satisfy the constraints the problem possesses no solution.

\textsuperscript{11}Hamilton's principle with fixed $t_0, t_1, q(t_0)$ and $\dot{q}(t_0)$ has been studied in \cite{5}. It is shown that the Lagrangian has to be modified to $K = -\frac{1}{2}q^T M(q)\dot{q}(t - t_1) - q^T Kq(t - t_1) + 2F^T q(t - t_1)$ in the action integral, where the last two terms account for the potential energy.
b) Search for \( m Q \tilde{I}(Q) = \min_{Q} \tilde{I}(q), q \in Q \), with \( \tilde{I}(q) = \int_{t_0}^{t_1} \hat{L}(q, \dot{q}, \tau) d\tau \), where \( \hat{L} \) is a suitably modified Lagrangian function.

In other words, one can a priori either modify the set of curves within which the action is to be minimized, without modifying the Lagrangian, or modify the Lagrangian at once\(^{(12)}\). In the sequel we shall describe two solutions that have been proposed by Kozlov and Treshchev [292] and Buttazzo and Percivale [85]. The first one follows path a), whereas the second one rather follows path b). These studies do not aim at showing existence of a minimizing curve. The first one [292] proves that in a modified space of admissible curves, the classical action is extremal for the motion of the system (i.e. for the motion that satisfies Hamilton's principle outside the impacts, and shock conditions at the impact time). The second one [85] proves that the motion of a rigid problem is an extremal of a modified action, which contains the singular percussion measure.

### 3.4.2 Modified set of curves

Let us describe the study proposed in [292]. This way of proceeding is closer to the usual formulation for smooth motions that one can find in mechanical textbooks than the mathematical work of [85]. This is the reason why we introduce it first.

Let us consider a \( n \)-degree-of-freedom system, with a unilateral constraint \( f(q) \geq 0 \). Assume that the shocks are perfectly elastic, i.e. \( \dot{q}(t_k^+) = M(q) \dot{q}(t_k^-) \). Let us consider some motion \( q_0(t) : [t_1, t_2] \rightarrow \mathbb{R}^n \), with \( f(q_0(t)) > 0 \) for \( t \in [t_1, t_2] \setminus \{t_0\} \), \( f(q_0(t_0)) = 0 \).

**Remark 3.1** Note that if kinetic energy is conserved at the impacts, then the impact times satisfy \( t_{k+1} > t_k + \delta \), for some \( \delta > 0 \) depending on initial data \( q(t_1), \dot{q}(t_1) \). Hence one can consider without loss of generality only one impact on the interval \( [t_1, t_2] \), chosen sufficiently small. If there was some energy loss at \( t_k \), such an assumption would not be possible due to the eventual accumulation point of the impact times sequence \( \{t_k\} \) (for any interval \( [t_1, t_2] \), there exists initial data at \( t_1 \) and external forces such that there is an infinity of rebounds in \( [t_1, t_2] \)).

Let us now consider the set of varied curves \( q_\alpha(t) : [t_1, t_2] \rightarrow \mathbb{R}^n \), with \( \alpha \in (-\varepsilon, \varepsilon) \) and

- \( q_\alpha(t_1) = q_0(t_1) \), \( q_\alpha(t_2) = q_0(t_2) \).
- \( q_\alpha(t) \) is a smooth function of \( \alpha \) and \( t \) in \( (-\varepsilon, \varepsilon) \times [t_1, t_2] \).
- \( f(q_\alpha(t)) = 0 \), where \( t_\alpha : (-\varepsilon, \varepsilon) \rightarrow [t_1, t_2] \) is a smooth function of the parameter \( \alpha \).

\(^{(12)}\)In a and b, one may replace \( \min I(q) \) by \( \text{extr} I(q) \). Indeed searching for the minimizing curve is very hard even in the non-constrained case. In general one finds the Euler-Lagrange equations which are only necessary conditions to be satisfied by the extremalizing curve. Whether or not these curves define a minimum point of the action is another problem.
3.4. HAMILTON'S PRINCIPLE AND UNILATERAL CONSTRAINTS

Kozlov and Treshchev choose the integral action

$$I(\alpha) = \int_{t_1}^{t_2} L(\dot{q}_\alpha(t), q_\alpha(t)) dt$$

(3.19)

where $L(\dot{q}, q)$ is the Lagrangian of the system. It is therefore clear now that the Lagrangian function is not modified, but that the set of varied curves is changed to curves with eventual discontinuous derivatives at the times $t_\alpha$. It is note worthy that both the curves and the impact times are varied, and that the varied curves attain the constraint. This is illustrated in figure 3.1. A first necessary step is to compute the variation of the action. The following is true

**Lemma 3.1 (Kozlov, Treshchev [292])** The variation of the action in (3.19) is given by

$$\delta I(0) = \frac{dI}{d\alpha}(0) = a_\alpha(q_0(t_0)) M(q(t_0)) \frac{dq_\alpha(t_0)}{d\alpha}(0)$$

$$- \frac{1}{2} \left[ \dot{q}_\alpha(t_0)^T M(q(t_0)) \dot{q}_\alpha(t_0) - \dot{q}_0(t_0^+)^T M(q(t_0)) \dot{q}_0(t_0^+) \right] \frac{dq_\alpha}{d\alpha}(0)$$

$$+ \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \left( \frac{\partial L}{\partial q} \right) \right]_{q=q_0} \frac{\partial q_\alpha(t)}{\partial \alpha}(0) dt$$

(3.20)

Let us note that $q_\alpha(t_\alpha), q_\alpha(t)$ and $t_\alpha$ are to be considered as functions of $\alpha$, and their derivatives are calculated at $\alpha = 0$. Note also that the second term in the variation of the action is in fact the kinetic energy loss at impact at time $t_0$ (if $q_0$ represents a motion of the system). The proof of lemma 3.1 can be found in [292] p.12. The next step is to prove that if $q_0$ is a motion of the system, then it is an extremal of the action, and vice-versa. It is proved in [292] that the following result is true

**Lemma 3.2 ([292])** The curve $q_0(\cdot)$ is a motion of the dynamical system with Lagrangian $L(\dot{q}, q)$, unilateral constraint $f(q) \geq 0$ and elastic collisions, if and only if $\delta I(0) = 0$, i.e. $q_0$ is a stationary point of the action integral.
Although the complete proof can be found in the book by Kozlov and Treshchev, let us give a sketch of it, for the sake of completeness of the exposition. Necessity (i.e. \( q_0 \) is a motion \( \Rightarrow \delta I(0) = 0 \)) follows mainly from three facts:

i) The kinetic energy loss at \( t_0 \) is zero (hence the second term in \( \delta I(0) \) is zero as well);

ii) The term \( \frac{d[q_0(t_0)]}{da} \) is orthogonal (in the kinetic metric sense) to \( \nabla_q f(q_0) \), whereas \( \sigma_0(t_0) \) is orthogonal to the surface \( f(q) = 0 \) at \( q_0 \). These results can be easily proved using the particular decomposition of the generalized velocity that is described in chapter 6, section 6.2. Indeed one can show that

\[
\sigma_0(t_0) = (\dot{q}_{\text{norm}}(t_0^+) - \dot{q}_{\text{norm}}(t_0^-)) n_q,
\]

with \( n_q \) defined in (6.2). Also since by assumption \( f(q_\alpha(t_\alpha)) = 0 \) for all \( \alpha \in (-\varepsilon, \varepsilon) \), it follows that \( \frac{\partial I}{\partial q} \frac{d[q_\alpha(t_\alpha)]}{da} = 0 \) for all \( \alpha \) as well, in particular for \( \alpha = 0 \);

iii) Since \( q_0 \) is a motion, the third term in \( \delta I(0) \) is clearly zero.

Sufficiency \( (\delta I(0) = 0 \Rightarrow q_0 \) is a motion) can be shown noting that \( \delta I(0) \) is zero for any type of variation in the family of varied curves \( q_\alpha(\cdot) \). Hence one can choose various varied family of curves and prove in three steps that i) \( q_0 \) is a motion on \([t_1, t_2] \setminus \{t_0\} \), ii) \( \dot{q}_{\text{tang}} \) is continuous at \( t_0 \) (see chapter 6, section 6.2 for details on the definition of \( \dot{q}_{\text{tang}} \)), iii) \( \dot{q}_{\text{norm}}(t_0^+) = -\dot{q}_{\text{norm}}(t_0^-) \).

### 3.4.3 Modified Lagrangian function

Let us choose now path 'b. It remains now to examine how the integral action whose extremals are the solution of the impact problem can be written. As suggested by the studies presented in chapter 2, the perfectly rigid case can be regarded as the limit of sequences of continuous-dynamics problems \( P_n \). Problems \( P_n \) are typically problems where the integrand of the integral action possesses different values depending on whether a given function \( f(q) \) is positive, zero or negative.

An interesting formulation in direction 'b is the one in [85] [427] [428]. We have already considered such approaches in chapter 2. These authors consider approximating variational problems \( P_n \) with Lagrangian

\[
L(q_n, \dot{q}_n) = T(q_n, \dot{q}_n) - U(q_n) - \alpha_n(f(q_n))
\]

where the last term accounts for the potential elastic energy when there is contact and will be defined below. The limiting or bounce problem \( P \) has Lagrangian

\[
L(q, \dot{q}) = T(q, \dot{q}) - U(q) + f(q)\mu,
\]

where the unilateral constraint is \( \mathbb{R} \geq f(q) \geq 0 \), and \( \mu \) is a bounded positive measure that represents the contact percussion. The problem is stated as follows:

**Problem 3.1 (Buttazzo, Percivale [85])** Let us consider a Lagrangian mechanical system, with \( V \) the system's configuration space. \( q(t) \in V \) is the generalized position vector, subject to the constraint \( f(q) \geq 0 \), and to a potential \( U(q) \). The kinetic metric of the system defines a scalar product on its tangent space at every
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$q$, $V$ is assumed to be be an $n$-dimensional manifold of class $C^3$, without boundary. Let $f : V \to \mathbb{R}$ be a function of class $C^3$ such that $\nabla_q f(q)$ is not zero on the set $\{ q \in V : f(q) = 0 \}$. Given $T > 0$, $\text{Lip}(0, T; V)$ denotes the space of Lipschitz continuous functions from $[0, T]$ into $V$. $L_1$ is the space of $L_1[0, T]$-bounded functions, twice differentiable. Then a pair $(U, q) \in L_1 \times \text{Lip}$ solves the bounce problem $P$ (or equivalently $q$ is a solution of $P$ with potential $U(t, q)$) if

i) $f(q(t)) \geq 0$ for every $t \in [0, T]$.

ii) There exists a finite positive measure $\mu$ on $(0, T)$ such that $q$ is an extremal for the functional

$$ F(q) = \int_0^T \left[ \frac{1}{2} q^T M(q(t)) \dot{q} - U(q(t)) \right] dt + \int_0^T f(q(t)) d\mu $$

and the support of $\mu$ satisfies $\text{supp}(\mu) \subseteq \{ t \in [0, T] : f(q(t)) = 0 \}$.

iii) for every $t_1, t_2 \in [0, T]$ the following energy relation holds

$$ 2 \int_{t_1}^{t_2} \nabla_q U(q(t))^T \dot{q}(t) dt = \dot{q}(t_2^+)^T M(q(t_2)) \dot{q}(t_2) - \dot{q}(t_1^+) M(q(t_1)) \dot{q}(t_1) $$

$$ = \dot{q}(t_2^+)^T M(q(t_2)) \dot{q}(t_2) - \dot{q}(t_1^-) M(q(t_1)) \dot{q}(t_1) $$

$$(3.23)$$

Similarly to the problem 2.1, i is the unilateral constraint. iii means that the system is conservative (or passive, or lossless in control theory language), during smooth motions and at the collisions with $f(q) = 0$: If $t_2$ is a collision time, then one deduces from iii that the kinetic energy loss is $T(t_2^+) = T(t_2^-)$, by taking $t_2$ during a smooth motion period. In other words $[427]$, the function $T : t \to \dot{q}(t)^T M(q(t)) \dot{q}(t)$ is continuous. From ii, a solution $q(t)$ must satisfy

$$ M(q) \ddot{q} + C(q, \dot{q}) + \frac{\partial U}{\partial q}(q) = \frac{\partial f}{\partial q} h $$

$$(3.24)$$

The last integral term in (3.22) can be written as $\int_0^T f(q(t)) \dot{h}$, where using our preceding notations $h(t)$ denotes the "impact" step-function such that $dh = \dot{h} = \sigma_h(t_k) \delta_{tk}$ is a singular distribution. In other words $\sigma_h(t_k)$ is the density of $\mu$ with respect to the Dirac measure $\delta_{tk}$, see definition 1.2. $d\mu = dh$ is a shorthand for $\delta_{tk}$. As we shall see in the sweeping process formulation, the Lagrange equations of the system, which we wrote in (3.24) as an equality between functions, is an equality of measures that should be written as

$$ M(q) \ddot{q} + \left[ C(q, \dot{q}) + \frac{\partial U}{\partial q}(q) \right] dt = \frac{\partial f}{\partial q} d\mu $$

$$(3.25)$$

where $d\dot{q}$ is the measure$^{13}$ defined by the generalized (or distributional) derivative of $\dot{q}$. From the developments in chapter 1 this is clearly equivalent outside the atoms

---

$^{13}$In chapter 1 we indicated that distributional derivatives of a function $f$ are sometimes denoted as $Df$. The notation $d\dot{q}$ is also used in nonsmooth dynamics to denote the measure associated to a function $RCLBV$ [381] and is called the differential measure of $f$. 

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of $\mu$ and of $d\dot{q}$ to the classical Lagrange equations. The formulation as in (3.25) is at the base of nonsmooth dynamics.

Similarly as for problem 2.2, the solutions of the limit problem are such that $\dot{q} \in RCLBV$.

**Remark 3.2** Time dependent potential terms $U(t, q(t))$ can also be considered.

The analysis proceeds to prove that any limit of a sequence of solutions of $\mathcal{P}_n$ converges towards a solution of $\mathcal{P}$, and that any solution of $\mathcal{P}$ is the limit of such an approximating sequence. The approximating sequences are chosen as the pairs $(U, q_n) \in L_1 \times \text{Lip}$ such that $q_n$ is the extremal of the functional

$$F_n(q_n) = \int_0^T L(q_n(t), \dot{q}_n(t)) dt$$

where $L(q_n, \dot{q}_n)$ is defined in (3.21) and the function $\alpha_n$ satisfies

- $\alpha_n(x) = \int_x^0 \psi_n(y) dy$, $\lim_{n \to \infty, x \to 0^-} \frac{\psi_n(x)}{\alpha_n(x)} = +\infty$
- $\psi_n \to +\infty$ uniformly on any compact subset of $(-\infty, 0)$
- $\psi_n$ is continuous, $\psi_n \geq 0$, $\psi_n(x) = 0$ if $x \geq 0$.

These conditions are similar to those in problem 2.1. Notice that $F_n(q_n)$ in (3.26) and the action $I_\alpha$ in (3.19), subsection 3.4.2, are of different natures. The first one is a sequence of actions (hence countable set), whereas the second one is a family of actions corresponding to a family of varied curves ($\alpha$ is a real taking values in an open interval).

Convergence of the approximating problems $\mathcal{P}_n$ towards $\mathcal{P}$ is understood in the sense of $\Gamma$-convergence [118]. The initial data of the problem are taken as described in chapter 2, theorem 2.7. The main result of Buttazzo and Percivale is the following:

let us define the sets

$$A(\tau_0) = \{(b, U, q) : (U, q) \in E, \mathcal{T}(\tau_0, q) = b\}$$

and

$$A_n(\tau_0) = \{(b_n, U_n, q_n) : (U_n, q_n) \in E_n, \mathcal{T}_n(\tau_0, q_n) = b_n\}$$

where the sets $E, E_n, \mathcal{T}, \mathcal{T}_n$ are the same as in theorem 2.7. Then

**Theorem 3.2** ([85]) for every $\tau_0 \in [0, T]$, $(b, U, q) \in A(\tau_0)$ if and only if there exist $b_n \to b$, $U_n \to U$, $q_n \to q$ such that $(b_n, U_n, q_n) \in A_n(\tau_0)$ for all $n$ large enough.

In other words, it is proved that if the problems $\mathcal{P}_n$ possess suitable solutions, these solutions converge towards limits which are in turn solution of the limit problem $\mathcal{P}$. Vice-versa, if a problem $\mathcal{P}$ possesses a solution, then there exists a sequence of approximating problems $\mathcal{P}_n$ whose solutions converge towards that of $\mathcal{P}$.

Uniqueness of solutions of the Cauchy problem 3.1 has been presented in theorem 2.7.
Remark 3.3 The works in [85] [427] [428] [292] are restricted to conservative systems. However it is known from the inverse problem in dynamics that one can associate a Lagrangian function to dissipative systems, like for instance the mass+spring+damper system considered above. A possible research work is extension of these studies to dissipative problems $\mathcal{P}_n$, by considering such modified Lagrangians (In the above simple case, we get $L(x, \dot{x}) = \left(\frac{1}{2}m\dot{x}_n^2 - \frac{1}{2}k_n x_n^2 \right) \exp \left(\frac{t_n}{m}\right)$). Also using the coordinate change $(q_1, ..., q_n) \rightarrow (q_1, ..., q_{n-1}, f(q))$, the Lagrangian of $\mathcal{P}$ could be chosen as $L(q, \dot{q}) = T(q, \dot{q}) - U(q) - \int q_r h dq$, with $h$ the density with respect to the Lebesgue’s measure of $\mu$ in (3.22), and $\int h = (0, ..., 0, 1)$. It is clear however that the addition of damping will yield possible finite accumulation points and complicate the convergence and existence studies. The existence result proved in [416], see theorem 2.1, indicates that results in this direction should be possible.

3.4.4 Additional comments and studies

A study on variational problems where the integral action has an integrand which varies can be found e.g. in [166]: the solution curves (extremaloids in the language of [166]) will in general possess refraction or reflection corners that correspond to jumps in the velocity; Grafinkel [166] proposes a systematic procedure based on necessary and sufficient conditions that the minimizing curves must satisfy to construct explicitly such a curve. However it is not clear how we should use the work in [166] to study the transition between the compliant and rigid cases. Variational problems with unilateral constraints are studied in [165] [183] [435] but the solutions possess a continuous first derivative, so that impact dynamics are excluded from these studies. Brief ideas and a sketch of approach are proposed in [438] §3.8, about the method of penalizing functions (denoted method of elastic stops in [438]). A pioneering work can also be found in Valentine [549], where so-called slack-variables are introduced.

Many mathematical works have been devoted to study existence of optimal solutions in calculus of variation for systems whose state is subject to a unilateral constraint, see e.g. [577] and references therein.

Murray [389] considers a classical Bolza variational problem, i.e. minimization of $I(q) = l(q(t_0), q(t_1)) + \int_{t_0}^{t_1} L(q, q, t) dt$, with $L(t, q, v)$ convex in $v$ for each $t, q$ and $l(\cdot, \cdot)$ lower semicontinuous. The trajectories $q(t)$ are allowed to possess discontinuities. Hence the velocity may contain singular distributions. Since it appears impossible to derive existence results with discontinuous curves for the action $I(q)$, a modified integral action $\bar{I}(q)$ that incorporates in some sense the jumps and singular measures terms is used in [389]. It has the feature to reduce to $I(q)$ when the trajectories are absolutely continuous. For the sake of comparison with the modified action used in [85], see problem 3.1, let us write the form of $\bar{I}(q)$:

$$\bar{I}(q) = I(q) + \int_{t_0}^{t_1} r(t, q, \xi) d\mu + \sum_{t_0 \leq t \leq t_1} g(t, q(t_k^-), q(t_k^+))$$ (3.29)

where $r(t, q, \xi) = \lim_{\lambda \rightarrow +\infty} \frac{L(t, q, v + \lambda \xi) - L(t, q, v)}{\lambda}$, is called the recession function, and
is independent of $v$. Due to the fact that $q$ is taken as a bounded variation function, one has $dq = \dot{q}dt + \xi_n d\mu_n + \xi_a d\mu_a$, where $dt$ is the Lebesgue measure, $\dot{q}$ is the differentiable part of $q$, $\xi_n d\mu_n$ is the nonatomic singular part of $\dot{q}$, and $\xi_a d\mu_a$ is its atomic singular part, see appendix C. Finally $g(t,a,b) = \inf_{y \in A} \left\{ \int_0^1 r(t,y) dt : y(0) = a, y(1) = b \right\}$, where $A$ is the set of absolutely continuous functions on $[t_0,t_1]$. $g(t,a,a) = 0$ so that the third term in (3.29) involves summation over a countable number of instants since $q \in BV([t_0,t_1])$. The set $B$ of admissible curves chosen in [389] is that of equivalent classes of functions of bounded variation on $[t_0,t_1]$, and the considered convergence is the weak * convergence on $B$. Compactness of the level sets of $\tilde{T}(\cdot)$ is proved in [389], hence existence of a minimizing curve in $B$. The interesting point in the study of [389] is the modification of the integral action, with the addition of a nonsmooth term to handle the singularities of the minimizing curves. The trajectories considered in [389] are too "irregular" for an impact problem (for us $q(t)$ is absolutely continuous whereas the velocity is of bounded variation). It may correspond to some optimal characterization of Bressan’s hyperimpulsive systems (see subsection 1.2.2). But more smoothness is likely to provide less difficulty in the analysis.

Note also that singular distributions naturally arise in other kinds of optimization problems with unilateral constraints [109] [460]. Recently Amar and Mariconda [6] proved the existence of a solution to the problem $\min \left\{ \int_0^T f(q(t)) dt : q \in K \subset W^{1,p}(0,T), q(t) \notin \Gamma, q(0) = q_0 \notin \Gamma, q(T) = q_T \notin \Gamma \right\}$, where $\Gamma \subset \mathbb{R}^n$. The function $f : \mathbb{R}^n \to \mathbb{R}$ is supposed to be lower semicontinuous, not necessarily convex, and satisfies the growth conditions $c_1||q||^p - c_2 \leq f(q)$ if $p > 1$, $\Xi(||q||) - c_2 \leq f(q)$ if $p = 1$, $c_1 \geq 0$ and $c_2 \geq 0$, $\Xi : \mathbb{R}^+ \to \mathbb{R}^+$ is convex and lower semicontinuous, with $\lim_{r \to +\infty} \Xi(r) = +\infty$. The proof is based on the convexification of the problem (i.e. find out a convex problem whose solution is also a solution of the nonconvex original problem). $\Gamma$ is an open subset of $\mathbb{R}^n$ (hence equivalently $q$ can be assumed to evolve in a closed subspace of $\mathbb{R}^n$, i.e. the complement $C_\Gamma$ of $\Gamma$ in $\mathbb{R}^n$). Smoothness of $\partial C_\Gamma$ is not a priori supposed in [6] ($\Gamma$ is an open convex polygon $\subset \mathbb{R}^n$), so that the result may apply to the motion of a free particle subject to linear codimension $\geq 1$ unilateral constraints, with any energetical behaviour at the collisions. Then $f(q) = \frac{1}{2}m\dot{q}^2$ is the kinetic energy, and $f(\cdot)$ satisfies the growth condition for $p = 1$ and with $\Xi(r) = \frac{m}{2}r^2$, $c_2 = 0$. For the relationships between Sobolev spaces and functions of bounded variation, see appendix C. Since we are primarily interested in velocities of bounded variation in view of stability analysis and control, the existence result in [6] is not sufficient for us.

Ivanov and Markeev [230] apply the nonsmooth change of variable described in subsection 1.4.2 to a $n$-degree-of-freedom system with an ideal constraint $q_0 \geq 0$, and elastic reflections. They prove that in the new coordinates, the curve $q^*(\cdot) = (q_0^*(t), q_1(t), \ldots, q_n(t))$, with $q_0(t) = |q_0^*(t)|$, is an extremal of the action $\int_{t_1}^{t_2} L^*(q^*(t), \dot{q}^*(t), t) dt$. Then they write the dynamics of the transformed system under a Hamiltonian form and show on an example how this can be used to study
the stability of motion.

Finally let us note that Hamilton's principle for systems with unilateral constraints has also received attention in the works by Moreau [384] and Panagiotopoulos [413].

Unilateral constraints and optimal control

Many studies have been also devoted to the problem of optimal control either when some unilateral constraints are imposed on the state and/or the control, or directly in the framework of measure differential equations (MDE). The idea is here to use the close relationship between variational calculus and optimal control [608]: instead of minimizing \( I(q) \) with Lagrangian \( L(q, \dot{q}, t) \), one minimizes \( I(u) \), with Lagrangian \( L(q, u, t) \), subject to \( \dot{q} = u \). Then \( q(\cdot) \) is recovered by integration. Schmaedeke [477] considers a cost functional of the form \( C(u) = \int_{t_0}^{t_1} L(t, x(t), u(t)) dt \), together with a MDE \( Dx = f(t, x(t), u(t)) + G(t)Du \). Then the author proves the existence of an optimal control function \( u \) of bounded variation that minimizes \( C(u) \) (provided some assumptions are made, see [477] theorem 12). In this setting, let us mention the studies in [45] (restricted to constraints of the form \( f(q, u, t) \geq 0 \)) and in [46] (where constraints \( f(t, q) \geq 0 \) are considered), and the works of the Russian school, on extension of Pontryagin's principle to systems with unilateral state constraints, see e.g. [20] and references therein. A common fact to all those results is that the solutions possess discontinuities (corners) when the boundary of the domain is attained transversely. In particular, the existence results in [20] are formulated as a Mayer problem for systems as in (0.1), but apply to integral-type functionals.
Chapter 4

Two bodies colliding

The foregoing chapters are devoted to investigate the mathematical nature of impact dynamics. Let us note at once that a dynamical problem with the smooth dynamics and the unilateral constraints is not complete, since one needs the knowledge of the behaviour of the system on the constraints surface to investigate global motion. For instance some authors assume that post-impact velocities are tangential to the constraints (in the sweeping process formulation that we describe later), or that there is no energy loss at impacts (see the mathematical studies in [84] [85] [93] [94] [427] [428] [476]). In fact, the behaviour at the impact is generally modeled by a set of coefficients (not necessarily constant) that relate post and pre-collisions values. In this chapter we first investigate in some detail the dynamics of two rigid bodies that collide. Then we review the existing studies on macroscopic models of the shock phenomenon.

4.1 Dynamical equations of two rigid bodies colliding

4.1.1 General considerations

In order to make this clear, let us first write down the dynamical equations of two rigid bodies in a 3-dimensional ambient (or "real-world") Cartesian space, in the framework of systems with unilateral constraints.

Let each body be represented by a set of generalized coordinates \( q_i = \begin{bmatrix} X_i \\ \xi_i \end{bmatrix} \), where \( X_i \in \mathbb{R}^3 \) represents the gravity center coordinates in some Galilean frame \( \mathcal{G}_i \), and \( \xi_i \in \mathbb{R}^3 \) is a set of three Euler angles for the bodies orientation. We denote the positive definite inertia matrix of each body by \( M_i(q_i) \in \mathbb{R}^{6 \times 6} \). The kinetic energy of each body is given as

\[
2T_i = m_i \dot{X}_i^T \dot{X}_i + \Omega_i^T I_i \Omega_i
\]  

(4.1)
where $I_i$ is the inertia tensor expressed in a suitable coordinate frame, and $\Omega_i \in \mathbb{R}^3$ is the instantaneous angular velocity. (1)

$\Omega_i$ is related to $\xi_i$ by $\Omega_i = J_{\xi_i}(\xi_i)\dot{\xi}_i$, for some locally nonsingular matrix $J_{\xi_i}(\xi_i)$. It is well-known that $J_{\xi_i}(\xi_i)$ is not in general a Jacobian. When $\Omega_i$ is expressed in a coordinate frame attached to the body, and when $\xi_i$ is a set of 3 $z, x, z$ Euler angles $\Phi_i, \theta_i, \psi_i$, then the so-called Olinde-Rodrigues [529] formula sets that:

$$
J_{\xi_i} = \begin{pmatrix}
\sin(\psi_i)\sin(\theta_i) & \cos(\psi_i) & 0 \\
\sin(\theta_i)\cos(\psi_i) & -\sin(\psi_i) & 0 \\
\cos(\theta_i) & 0 & 1
\end{pmatrix}
$$

whose determinant vanishes at $\theta = k\pi$, $k \in \mathbb{Z}$. We thus obtain that the inertia matrix is given by

$$
M_i(q_i) = \text{diag}(m_i) \times J_{\xi_i}
$$

This allows us to write down the dynamical equations of each body as (we drop the arguments for convenience)

$$
M_i\ddot{q}_i + C_i\dot{q}_i + g_i = Q_i
$$

which is similar to the equation derived in example 1.3.

**Remark 4.1** Recall that if $\Omega_i$ is expressed in a frame fixed with respect to body $i$ and composed of the principal axes, and centered either at the gravity center $G_i$ or at a fixed point, then the so-called Euler equations are given by [16]

$$
\frac{d}{dt}(M_{a,i}) = M_{a,i} \times \Omega_i
$$

where $M_{a,i}$ denotes the angular momentum of body $i$, or, since $M_{a,i} = I_i\Omega_i$ in that frame

$$
I_i\frac{d}{dt}(\Omega_i) = I_i\Omega_i \times \Omega_i
$$

These equations relate the angular momentum variation to the instantaneous velocity. If some torques $N_i$ act on the body then we get

$$
I_i\frac{d}{dt}(\Omega_i) = I_i\Omega_i \times \Omega_i + N_i
$$

1Recall that, strictly speaking, such vector contains the three independent components of the skew-symmetric tangent operator of the rotation matrices of the bodies, belonging to the group $SO(3)$ of spatial rotation matrices [323] §2.3. In other words, if the displacement, or the flow, is represented by $\varphi_t(x_0) = A(t)x_0$ with $A(t) \in SO(3)$, then $\dot{\varphi}_t(x_0) = A(t)A^{-1}(t)\dot{\varphi}_t(x_0)$, i.e. $D_{x_0}\dot{\varphi}_t = [AA^{-1}](t)$ is the tangent map (or operator) of the flow at $x_0$. It is known that $AA^{-1}$ for $A \in SO(n)$ is a skew-symmetric $n \times n$ matrix, whose $\frac{n(n-1)}{2}$ independent entries are the components of the "angular velocity" $\Omega \in \mathbb{R}^\frac{n(n-1)}{2}$.
4.1. DYNAMICAL EQUATIONS OF TWO RIGID BODIES COLLIDING

Now if the torques $N_i$ are equal to $p_i \delta_{t_k}$, we get

$$I_i \sigma_{\Omega_i}(t_k) = p_i$$

(4.8)

which provides the instantaneous velocity jump at $t = t_k$. Also it follows that

$$\sigma_{\xi_i}(t_k) = J_{\xi_i}^{-1} \sigma_{\Omega_i}(t_k).$$

\[\nabla\nabla\]

Now since the bodies cannot interpenetrate, there is in addition to the two dynamical equations in (4.4) a unilateral constraint expressing that the distance between the bodies must be positive, i.e.

$$f(q_1, q_2) \geq 0$$

(4.9)

for some function $f : \mathbb{R}^{12} \rightarrow \mathbb{R}$, assumed to be smooth enough. We also assume that there is a unique possible contact point between bodies 1 and 2, so that the surface $f(q_1, q_2) = 0$ is of codimension 1 $^2$. As we shall see in the next chapters, this is an important assumption because it means that only simple impacts are to be considered between the two bodies. If two points were to be reached simultaneously, one would have to treat a multiple impact. This is a much more difficult task, from all points of view (model, mathematical properties of the dynamical equations), as we shall see in chapters 5 and 6.

For instance in the case of two spheres, then the configuration spaces have respective dimensions 3 (the center of gravity position), and the constraint is simply $||G_1G_2||^2 \geq (R_1 + R_2)^2$, or $f(X_1, X_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - (R_1 + R_2)^2 \geq 0$. For convenience we may introduce a new set of generalized coordinates $q_1 = x_1$, $q_2 = y_1$, $q_3 = z_1$, $q_4 = x_1 - x_2$, $q_5 = y_1 - y_2$, $q_6 = z_1 - z_2$. Then we get

$$\nabla_q f(q)^T = (0, 0, 0, 2q_4, 2q_5, 2q_6) \in \text{span} \{e_4, e_5, e_6\},$$

where $e_i$ is the $i$th unit vector of $\mathbb{R}^6$.

When the two bodies make contact, then $f(q_1, q_2) = 0$ and there is a common point $A = A_1 \in B_1 = A_2 \in B_2$ at which they touch each other $^3$. We suppose that the bodies' surfaces are smooth enough so that a common normal direction $n \in \mathbb{R}^3$ can be defined at $A$. At $A$ we can associate two orthonormal local frames $\mathcal{L}_1 = (n_1, t_{11}, t_{12})$ and $\mathcal{L}_2 = (n_2, t_{21}, t_{22})$, where $n_1 = -n_2$ are colinear to $n$ and point outwards the respective bodies, whereas $n_1 \times t_{11} = t_{12}$, $t_{11} \times t_{12} = n_1$, see figure 4.1. This frames are not attached to the bodies, but are just defined so that at the shock instant, $A$ coincides with the contact point, see subsection 4.1.3 for the motivation for this choice. Now we have

$$V_{A_i} = V_{G_i} + A_i G_i \times \Omega_i$$

(4.10)

$^2$In reality it is clear that contact always exists over a finite surface surrounding what we call the contact point. The rigid body model is an approximation of this surface to a point.

$^3$A is the virtual point that coincides with $A_1 \in B_1$ and $A_2 \in B_2$ at which contact is established.
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Figure 4.1: Local frames at the contact point.

where all the vectors are expressed in \((A, \mathcal{L}_i)\) (for the sake of simplicity, we keep the notation \(\Omega_i = \begin{pmatrix} \omega_{i1} \\ \omega_{i2} \\ \omega_{i3} \end{pmatrix}\) for the angular velocity). In other words, the twist calculated at \(G_i\) is given by \(T_{G_i} = \begin{bmatrix} 0 & I_3 \\ I_3 & 0 \end{bmatrix} J \dot{\Omega}_i = \begin{bmatrix} \Omega_i \\ V_{G_i} \end{bmatrix}\), with \(J = \begin{bmatrix} I_3 & 0 \\ 0 & J_{\xi_i} \end{bmatrix}\), and calculated at \(A_i\) by \(T_{A_i} = \begin{bmatrix} \Omega_i \\ V_{A_i} \end{bmatrix}\). (4) These twists can be expressed in the frames \((A, \mathcal{L}_i)\) defined above, where in particular \(V_{A_i} = v_{i,n} n_i + v_{i,t_1} t_1 + v_{i,t_2} t_2\). It is preferable in general to choose either \(\mathcal{L}_1\) or \(\mathcal{L}_2\) and to express all quantities in a unique frame. In the following we choose without loss of generality the frame \((A, \mathcal{L}_1)\). Hence \(v_{r,n} = v_{1,n} - v_{2,n}\) is the relative normal velocity between both bodies (when the bodies detach, it is assumed that \(\|A_1A_2\| = \min_{A_1 \in B_1, A_2 \in B_2} \|A_1A_2\|\)). Now we shall write for simplicity \(V_{A_i} = v_{i,n} n_i + v_{i,t_1} t_1 + v_{i,t_2} t_2\). We can now write the transformation between the twists expressed in the local frames and the generalized velocities as

\[
T_{A_i} = \mathcal{M}_i \dot{\Omega}_i
\]

(4.11)

for some (locally) nonsingular transformation matrix \(\mathcal{M}_i \in \mathbb{R}^{6 \times 6}\). If the transformation matrix from \(G\) to \(\mathcal{L}_1\) is \(T_1 \in \mathbb{R}^3\), and if \(A_iG_i = \begin{pmatrix} r_{1i} \\ r_{2i} \\ r_{3i} \end{pmatrix}\) (5), then it follows

\[\text{Recall [323] that the twist of a rigid body is nothing else but the components of the tangent operator of the group } H(4) \text{ of } 4 \times 4 \text{ homogeneous transformations of the form } T(t) = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}, \text{ where } R \in \mathbb{R}^{3 \times 3} \text{ is a rotation matrix, and } d \in \mathbb{R}^{3} \text{ represents linear displacements. The tangent operator is given by } \dot{T}(t)T^{-1}(t) = \begin{bmatrix} RR^{-1} & \dot{V} \\ 0 & 0 \end{bmatrix}, \text{ where the matrix } RR^{-1} \in \mathbb{R}^{3 \times 3} \text{ is skew-symmetric with 3 independent entries, and the twist (a screw) is given as a 6-dimensional vector } (\Omega, V). \text{ The components of } \Omega \text{ are the independent entries of } \Omega.\]

\[\text{It may for instance be assumed that } r_{ji} = r_{ji}(\xi_i), \text{ } j = 1, 2, 3, \text{ if the bodies' shape lends itself to analytic description.}\]
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From (4.10) that $\mathcal{M}_i$ is given by

$$
\mathcal{M}_i = \begin{bmatrix}
0 & T_1 J_i \\
T_1 & R_i T_1 J_i
\end{bmatrix}
$$

(4.12)

where $R_i = \begin{pmatrix}
0 & -r_{3i} & r_{2i} \\
r_{3i} & 0 & -r_{1i} \\
r_{2i} & r_{1i} & 0
\end{pmatrix}$. Due to the way we have written Varignon's formula in (4.10) and (4.11), $R_i$ is not a skew-symmetric tangent operator of $SO(3)$.

Clearly also the wrench of external actions acting on body $B_i$, expressed in the same frame $(A, \mathcal{L}_1)$, takes on the form

$$
\mathcal{W}_{A_i} = \begin{bmatrix}
F_i \\
C_{A_i}
\end{bmatrix}
$$

(4.13)

where $F_i \in \mathbb{R}^3$ is the resulting force, and $C_{A_i} \in \mathbb{R}^3$ the torque applied to body $i$ at $A_i$. Recall that the wrench can be written in its contravariant form as in (4.13) (in which case it is simply a vector of the linear space $\mathbb{R}^3$), or in its covariant form

$$
\mathcal{W}_{A_i}^* = \begin{bmatrix}
C_{A_i} \\
F_i
\end{bmatrix},
$$

in which case it belongs to the dual space to that linear space (which is the space of the body's twist). Also the scalar product of the twist and the wrench $\mathcal{W}_{A_i}^*$ is an invariant, and represents the power of the forces and torques acting on the body. When this scalar product is zero, then the twist and the wrench are said to be reciprocal. This is the case when the constraints are frictionless and when the velocities are compatible with the constraints. Then the normal (force) subspace and the tangential (velocities) subspace at $A$ are dual subspaces.

The following relationship relates the covariant components of the contact interaction force $Q_i$ to the real-world interaction wrench $\mathcal{W}_{A_i}^*$, as

$$
Q_i = \mathcal{M}_i^T \mathcal{W}_{A_i}^*
$$

(4.14)

Remark 4.2 In case of a configuration space of dimension smaller than 6, all vectors and matrices possess adapted reduced dimensions. Since we deal with bodies, the maximal dimension of the configuration space is 6 and equals that of the real-world values (twists and wrenches). In case of manipulators, it may happen that the number of degrees of freedom $n$ is larger than 6, hence $\mathcal{M}_i$ is rectangular and $\in \mathbb{R}^{6 \times n}$. More generally, the relationships between "real-world" (or task-space in robotics language [607]) velocities $\dot{x}$, and generalized velocities $\dot{q}$ can be summarized as shown in figure 4.2. In general $\dim(\dot{q}) \geq \dim(\dot{x})$ so that the mapping $J$ is an onto mapping, and $J^T$ is a one-to-one mapping. Recall that given a mapping $x \to y = Jx$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, the mapping $J$ is said to be onto if for all $y$, there exists an $x$ such that $y = Jx$ (6). It is said to be one-to-one if there exists $y$ such

Or equivalently $J : E \to F$ is said to be surjective if and only if $\text{Im}(J) = F$. 
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Figure 4.2: Velocity and reaction transformations.

that for all \( x \), \( Jx \neq y \) (\(^7\)). Note that \( Jx = \sum_{i=1}^{m} x_i J_i \), i.e. \( Jx \) is a linear combination of the columns \( J_i \). The space spanned by the \( J_i \)'s is therefore a subspace of (or equal to) \( \mathbb{R}^m \). If \( n > m \), clearly \( J \) is one-to-one. If \( m \geq n \), then if \( \text{rank}(J) = n \), \( J \) is onto. In case of a mechanical system subject to frictionless holonomic constraints \( f(q) = 0 \), \( f(q) \in \mathbb{R}^m \), then \( p_q = \frac{\partial f}{\partial q} \lambda = \sum_{i=1}^{m} \lambda_i \frac{\partial f_i}{\partial q} \), where \( \lambda \in \mathbb{R}^m \) are the Lagrange multipliers. If we associate some velocity \( \dot{x} \) to \( \lambda \) such that \( \dot{x} \) performs work on \( \lambda \), then \( \dot{x} = \frac{\partial f^T}{\partial q} \dot{q} \). If the constraints are independent (i.e. \( \text{rank}(\frac{\partial f}{\partial q}) = m \)) then \( \frac{\partial f^T}{\partial q} \) is onto whereas \( \frac{\partial f}{\partial q} \) is one-to-one.

Remark 4.3 The surfaces are assumed to be frictionless. This means that the hypersurface defined in the 12-dimensional configuration space \( f(q_1, q_2) = 0 \) is frictionless also. By the virtual work principle, it must be that when contact is established, all motions compatible with the constraints produce zero work, i.e. \( \delta q^T Q = 0 \), where \( Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \in \mathbb{R}^{12} \) and \( \delta q \) is an arbitrary virtual displacement of \( q \), compatible with the constraints. In other words, since such motion is tangential to the surface \( f(q_1, q_2) = 0 \), one must have \( Q = \lambda \nabla_q f(q_1, q_2) \) for some \( \lambda \in \mathbb{R} \). Hence from (4.14) we get

\[
\lambda \nabla_q f(q_1, q_2) = \begin{bmatrix} \mathcal{M}_1^T \mathcal{W}^*_{A_1} \\ \mathcal{M}_2^T \mathcal{W}^*_{A_2} \end{bmatrix} = \mathcal{M}^T \mathcal{W}^* \tag{4.15}
\]

with \( \mathcal{M}^T = \begin{bmatrix} \mathcal{M}_1^T & 0 \\ 0 & \mathcal{M}_2^T \end{bmatrix} \in \mathbb{R}^{12 \times 12} \), and \( \mathcal{W}^* = \begin{bmatrix} \mathcal{W}^*_{A_1} \\ \mathcal{W}^*_{A_2} \end{bmatrix} \in \mathbb{R}^{1 \times 12} \).

Furthermore note that the bodies are perfectly rigid and we shall not introduce any moment at the contact point since surfaces are frictionless. Hence \( C_{A_i} = 0 \) and \( F_i = F_{i,n} n_1 \) for some \( F_{i,n} \in \mathbb{R} \). The virtual work principle tells us that

\[
\delta q^T Q = \sum_{i=1}^{2} V_{A_i}^T F_i = \sum_{i=1}^{2} v_{i,n} F_{i,n} = 0 \tag{4.16}
\]

From the principle of mutual actions\(^8\), one has \( F_{1,n} = -F_{2,n} \), so that \( v_{r,n} F_{2,n} = 0 \). Note that the tangential components \( v_{i_1} \) and \( v_{i_2} \) may or may not be constrained.

\(^7\)Or equivalently \( J : E \rightarrow F \) is said to be injective if and only if \( \text{Ker}(J) = \{0\} \).

\(^8\)also known as Newton's third law, and supposed to be true for impulsive interactions.
depending on the type of contact (sliding or rolling without slipping). If contact is broken note that $F_{i,n} = 0$ so that (4.16) is still verified, although $v_{r,n}$ may be nonzero. We shall retrieve such formulation of contact laws when we deal with complementarity formulations, in which case some constraint on the relative normal acceleration $a_{2,n} = \dot{v}_{2,n}$ may be considered. Note also that when there is a shock the formulation in (4.16) makes no sense since $v_{r,n}$ is discontinuous at $t_k$ and $F_{2,n}$ is of the form $p_{2,n}\delta_{t_k}$ (we discussed this topic in chapter 2).

### 4.1.2 Relationships between real-world and generalized normal directions

Assume that on $[t_0 - \gamma, t_0]$ for some $\gamma > 0$ we have $f(q_1, q_2) = 0$. If contact is broken at $t = t_0$, then necessarily on $[t_0, t_0 + \delta]$, for some $\delta > 0$, we have $\dot{q}^T \nabla_q f(q_1, q_2) > 0$, i.e. the generalized velocity points outwards the constraint (i.e. inwards $q_1$) at least on a nonzero time interval. On $[t_0, t_0 + \delta]$ one must also have that the real-world velocities verify $V_{A_i}^T n_j \geq 0$, $i \neq j$, otherwise bodies would interpenetrate. This may be rewritten as $v_{r,n} = v_{1,n} - v_{2,n} \leq 0$, since $v_{1,n} \leq 0$ and $v_{2,n} \geq 0$. Hence we obtain the implications on $[t_0, t_0 + \delta]$

$$\dot{q}^T \nabla_q f(q_1, q_2) > 0 \Rightarrow V_{A_i}^T n_j \geq 0 \Rightarrow v_{r,n} \leq 0$$

for $i = 1, 2$. Now notice from (4.11) that $v_{i,n} = e_4^T M_i \dot{q}_i$, where $e_4 \in \mathbb{R}^{6 \times 1}$ is the fourth unit vector. Hence we can write

$$e_4^T M_2 \dot{q}_2 - e_4^T M_1 \dot{q}_1 \geq 0$$

or, noting that $V_{A_i} = E_3 M_i \dot{q}_i$,

$$\left\{\begin{array}{l}
n_1^T E_3 M_1 \dot{q}_1 \leq 0 \\
n_2^T E_3 M_2 \dot{q}_2 \geq 0
\end{array}\right.$$  

with $E_3 = [0 : I_3] \in \mathbb{R}^{2 \times 6}$. This in turn implies that $-q_1^T M_1^T E_3^T n_1 + q_2^T M_2^T E_3^T n_2 \geq 0$. More compactly

$$\dot{q}^T \tilde{M} n_1 \geq 0$$

with $\tilde{M} = \left[\begin{array}{cc}
-E_3 M_1 & E_3 M_2
\end{array}\right] \in \mathbb{R}^{3 \times 12}$. We finally get

$$\dot{q}^T \nabla_q f(q_1, q_2) > 0 \Rightarrow \dot{q}^T \tilde{M} n_1 \geq 0$$

from which we deduce the following

**Claim 4.1** For every $q \in \mathbb{R}^{12}$ such that $f(q_1, q_2) = 0$, then there exists a real $\lambda \geq 0$ such that

$$\tilde{M} n_1 = \lambda \nabla_q f(q_1, q_2)$$
This relationship relates the real world normal direction \( n \in \mathbb{R}^3 \) to the generalized normal direction \( \nabla_q f(q_1, q_2) \in \mathbb{R}^{12} \). It is a generalization of [383] proposition 3.1: if body 2 is fixed, then the total configuration space is 6-dimensional and (4.22) reduces to

\[
(E_3 \mathcal{M}_1)^T n_1 = -\lambda \nabla_{q_1} f(q_1) \tag{4.23}
\]

In section 6.3.1 an example is presented where \( n = 3 \) and \( n_1 \in \mathbb{R}^{2 \times 1} \).

### 4.1.3 Dynamical equations at collision times

Since we have assumed that the bodies' surfaces are frictionless, the interaction at the contact point is along \( n \in \mathbb{R}^3 \). Now as we have discussed previously, we deal with a system with a unilateral constraint. This implies some impulsive behaviour when the bodies make contact with nonzero (positive) approach normal relative velocity (we shall discuss later the case when some impulsive behaviour may occur with zero relative normal velocity). Here we get that each time contact is made at \( t = t_k \) with \( v_{r,n}(t_k^-) > 0 \), a shock occurs between both bodies at \( A \). From (4.4) (4.11) (4.13) and (4.14) one obtains the dynamical equations

\[
\dot{\mathbf{M}}_{n_k} \frac{d}{dt} \left( T_{A_k} \right) + \mathcal{M}_i^{-T} \left( C_i(q_i, \dot{q}_i) \mathcal{M}_i^{-1} T_{A_k} + g_i(q_i) \right) = \mathcal{M}_i^{-T} Q_i = \mathbf{W}_{A_k} \tag{4.24}
\]

where \( Q_i \) contains bounded as well as impulsive forces and the last equality comes from (4.14). Hence following the developments in chapter 1, one deduces that at a shock instant \( t_k \) the following algebraic equations are satisfied

\[
\begin{bmatrix}
\sigma_{\Omega_i}(t_k) \\
\sigma_{\mathbf{v}_{A_i}}(t_k)
\end{bmatrix}
= \begin{bmatrix}
0_{3 \times 1} \\
... \\
p_{i,n} \\
0 \\
0
\end{bmatrix} \tag{4.25}
\]

where the quantities are expressed in the local frame \( \mathcal{L}_i \), and it is assumed that there is neither impulsive moment at the contact point, nor Coulomb's friction. It is noteworthy that (4.25) is true independently of the fact that the frame \( \mathcal{L} \overset{\Delta}{=} (A, \mathcal{L}_1) \) used to express the dynamics is Galilean or not. In other words, if the used local frame is not Galilean, the velocity of a point \( M \) with respect to \( G \) expressed in \( G \) (the absolute velocity \( V_{M/G,G} \)) is equal to

\[
V_{M/G} = V(M/\mathcal{L}, \mathcal{G}) + V(A/\mathcal{G}, \mathcal{G}) + \Omega \times (OM - OA) \tag{4.26}
\]

where \( \Omega \) is the angular velocity of \( \mathcal{L} \) with respect to \( \mathcal{G} \), and \( O \) denotes the origin of \( \mathcal{G} \). The first term in the right-hand-side of (4.26) is the relative velocity, the second term is the velocity of motion of the moving coordinate system in \( \mathcal{G} \), the third one is the transferred velocity. All this is developed in [16] §26, 27. Hence if the motion of
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the local frame $\mathcal{L}$ with respect to $\mathcal{G}$ is smooth enough (9), one gets from (4.26) that

$$\sigma_{V_{\mathcal{M}/\mathcal{G}}}(t_k) = \sigma_{V_{\mathcal{M}/\mathcal{L},\mathcal{G}}}(t_k).$$

In other words, the so-called inertial (or fictitious) forces do not play any role in the velocity jump calculation, because such accelerations are merely bounded functions of time, and the jump of the absolute velocity is that of the relative velocity.

The matrix $\tilde{\mathbf{M}}_{A_i}$ is calculated as $\mathcal{M}_i^{-T}M_i(q_i)\mathcal{M}_i^{-1}$, where $\mathcal{M}_i$ is defined in (4.11) (4.12) and $\tilde{\mathbf{M}}_{A_i}$ can be calculated from (4.11) (4.12) and (4.3). First note that

$$\mathcal{M}_i^{-1} = \begin{bmatrix} -T_1^{-1}R_i & T_1^{-1} \\ (T_1J_i)^{-1} & 0 \end{bmatrix}$$

(4.27)

where $T_1, R_i$ and $J_i$ are defined in (4.12) and (4.2) respectively, and by assumption $\det(\mathcal{M}_i) = -\det(T_1^2J_i) \neq 0$. Hence the inertia matrix $\tilde{\mathbf{M}}_{A_i}$ takes the form

$$\tilde{\mathbf{M}}_{A_i} = \begin{bmatrix} m_iR_i^TT_1^{-1}T_1^{-1}R_i + T_1^{-TT_1}T_1^{-1} & -m_iR_i^TT_1^{-TT_1} \\ -m_iT_1^{-TT_1}T_1^{-1}R_i & m_iT_1^{-TT_1} \end{bmatrix}$$

(4.28)

From (4.25) and (4.28) it follows that

$$\sigma_{V_{A_i}}(t_k) = \begin{bmatrix} \frac{1}{m_i}T_i^TT_i^T + R_iT_i^{-TT_1}T_1^TR_i^T \end{bmatrix} P_i$$

(4.29)

whereas

$$\sigma_{\Omega_{A_i}}(t_k) = T_iI_i^{-1}T_i^TR_i^T P_i$$

(4.30)

with $P_i = \begin{bmatrix} p_{i,n} \\ p_{i,t_1} \\ p_{i,t_2} \end{bmatrix}$ ($p_{i,t_1} = \epsilon p_{i,t_2} = 0$ in (4.25) since friction or other tangential effects are neglected). If one conveniently chooses $\mathcal{G} = \mathcal{L}_1$ then $T_1 = I_3$ and the above formulas are simplified to

$$\tilde{\mathbf{M}}_{A_i} = \begin{bmatrix} m_iR_i^TR_i + I_3 & -m_iR_i^T \\ -m_iR_i & m_iI_3 \end{bmatrix}$$

(4.31)

and

$$\sigma_{V_{A_i}}(t_k) = \begin{bmatrix} \frac{1}{m_i}I_3 + R_iI_i^{-1}R_i^T \end{bmatrix} P_i$$

(4.32)

$$\sigma_{\Omega_{A_i}}(t_k) = I_i^{-1}R_i^TP_i$$

(4.33)

The inertia center $G_i$ velocity jump is given by

$$m_i\sigma_{\mathcal{G}_i}(t_k) = P_i$$

(4.34)

9Note that this implies in particular that the frame we have defined above to express the dynamics is not attached to body 1, since the velocity of body one undergoes discontinuities at $t_k$. It is just chosen so that its origin $A$ coincides with the contact point at the shock instant. But it may be mobile, with nonzero acceleration.
and the Euler angles derivatives jumps are given by

\[
\sigma_{\xi_i}(t_k) = J_{\xi_i}^{-1} T_{\xi_i}^{-1} R_{\xi_i}^T P_i \tag{4.35}
\]

The relationship in (4.25) relates the jump in the twist of body \(i\) to the percussion wrench at \(t = t_k\). Also recall that we have \(p_{1,n} = p_{2,n}\) from the mutual actions principle of Newton. The equality in (4.25) is to be compared with the one in (1.12) in example 1.3. It is nothing else than the generalization to 3-dimensional bodies of the equations of two particles colliding, moving on a line, which gives \(m_1 \sigma_{\xi_1}(t_k) = p_{12}\) and \(m_2 \sigma_{\xi_2}(t_k) = -p_{12}\) at the impact time. The equation \(m_1 \sigma_{\xi_1}(t_k) + m_2 \sigma_{\xi_2}(t_k) = 0\) is generalized to

\[
\begin{bmatrix}
\sigma_{\Omega_1}(t_k) \\
\sigma_{V_{A_1}}(t_k)
\end{bmatrix}
+ \begin{bmatrix}
\sigma_{\Omega_2}(t_k) \\
\sigma_{V_{A_2}}(t_k)
\end{bmatrix}
= 0 \tag{4.36}
\]

which is known as the linear and angular momenta conservation equations.

**Remark 4.4** The impact equations for two bodies colliding can be written in different ways, one of which is (4.36). In the particular planar case of two bodies colliding, the equation in (4.36) can be written as [279]

\[
\sigma_{\xi_1}(t_k) = \sigma_{\xi_2}(t_k) = m_1 \sigma_{\xi_1}(t_k) + m_2 \sigma_{\xi_2}(t_k) = 0 \tag{4.37}
\]

where \(x_i, y_i\) are body \(i\) gravity center \(G_i\) coordinates in the frame \((t_i, n_i)\) centered at the contact point \(A\), and

\[
\sigma_{M_{a,i}}(t_k) = 0, \quad i = 1, 2 \tag{4.38}
\]

where \(M_{a,i}\) is body \(i\) angular momentum computed at point \(O_i\), where \(O_i\) belongs to the axis \((A, n_i)\), and \(O_i\) has coordinates \((0, a_i)\) in the frame \((A, t_i, n_i)\). \(M_{a,i} = I_i \Omega_i + m_i AG_i \times V_{G_i} = I_i \Omega_i + m_i [x_i y_i - \hat{x}_i (y_i - a_i)]\). This in turn can be rewritten as

\[
I_i \sigma_{\Omega_i}(t_k) + m_i x_i \sigma_{\xi_1}(t_k) = 0, \quad i = 1, 2 \tag{4.39}
\]

\[\nabla \nabla\]

Let us note that there are 13 unknowns in this dynamical algebraic problem: 12 postimpact velocities (the \(\dot{q}_i\)'s or equivalently the \(\dot{\Omega}_i\)'s and the \(V_{A_i}\)'s components), and the percussion component \(p_n = p_{1,n} = -p_{2,n}\). For the moment we have only 12 equations given by (4.25) for \(i = 1, 2\). Clearly one additional equation is needed to render the impact problem solvable, i.e. calculate the postimpact velocities and the percussion vector. In fact as we shall see throughout the rest of this book, the whole problem of rigid body impact dynamics and of systems with unilateral constraints is to be able to compute these postimpact values. This is not always evident, especially when friction is considered. In the following we discuss the various ways proposed in the literature to model the collisions, i.e. in fact to associate to the dynamical problem suitable relationships that allow to calculate postimpact values.
4.2 Percussion laws

The basic and most widely used percussion law for frictionless shocks between rigid bodies is the so-called Newton's rule \(^{(10)}\). It relates the relative normal velocities after and before the shock as follows:

\[
e = \frac{v_{1,n}(t_k^+) - v_{2,n}(t_k^+)}{v_{1,n}(t_k^-) - v_{2,n}(t_k^-)} = \frac{v_{r,n}(t_k^+)}{v_{r,n}(t_k^-)}
\]  

(4.40)

where the kinematic coefficient \(e\) is \(0 < e < 1\). \(e\) is an experimental coefficient, and has a clear energetic meaning. For two particles colliding, one can easily show that the kinetic energy loss at impacts is given by \(^{[55]}\)

\[
T_{r}(t_k) = T(t_{k}^{+}) - T(t_{k}^{-}) = \frac{1}{2} \left( \frac{m_1 m_2}{m_1 + m_2} \right) (e^2 - 1) \left( v_{r,n}(t_{k}^{-}) \right)^2
\]  

(4.41)

Therefore the impact problem \(^{(11)}\) is always energetically consistent in this case since there cannot be a positive gain of energy at impact. The energetical considerations make it clear why \(e\) is less than unity to insure \(T_{r}(t_k) \leq 0\) \(^{(12)}\). Let us note that in general the kinetic energy loss is not an easy expression to obtain, due to the dynamical couplings between the various velocity variables, see (4.25).

**Remark 4.5** At some places of this book, we shall say that the notion of restitution coefficients is necessary to render the impact problem solvable. This is not entirely true. More exactly they are sufficient. Indeed one may argue that energy and momentum conservation laws may serve as well to solve the shock dynamical equations. It was indeed the underlying basic idea used for instance by Huygens when answering to a suggestion of the Royal Society of London in 1668, about impact dynamics (Huygens' work is recalled in \([292]\) Introduction). In the particular case of energy conservation at impacts, one may also directly use the equation derived from the kinetic energy conservation to prove some results, see the mathematical works in chapter 2, section 2.2. One may also describe the shocks via so-called complementarity formulations, like generalized dissipative shocks in the sweeping process formulation, see section 5.3. However as we shall see in chapter 6, energy and momentum laws are clearly not sufficient in general to solve the shock dynamics. Then the introduction of restitution coefficients may be of great help. Also the consideration of friction yields too complex collision process to allow a simple solution with energy and momentum laws. We shall also describe in chapter 5, section 5.3, alternative ways of modeling a shock process in the configuration space, which do

\(^{10}\)which is actually a conjecture: it is assumed that the impact process is represented by such a coefficient. This has to be proved one way or another.

\(^{11}\)More exactly: this impact model is always consistent.

\(^{12}\)Some authors \([104]\) \([510]\) define a restitution coefficient by considering the energy transferred into the "impacted" object as being lost, and by taking into account only the rebound velocity of the "impacting" mass. Hence an elastic collision without any kinetic energy loss has a restitution less than 1.
not make a priori reference to any restitution coefficients. Still one may argue that there exists a restitution law (or matrix) that corresponds to the said model, in the sense that the same postimpact velocities would be obtained with the restitution or with the original model.

The dependance of the restitution upon several parameters has been known and studied for a long time. An interesting survey has been done by Barmes [34], concerning collisions between spheres of same radii. It was concluded that the restitution coefficient depends mainly on the following physical effects:

- The relative approach velocity $v_{r,n}(0^-)$: $e$ decreases when $v_{r,n}(0)$ increases, [11]. More recent experimental studies can be found in [99] [224] [250] [527] [528] [534]. For instance at moderate contact velocities $v_{r,n}(0^-)$, Jonhson [250] finds the empirical law $e = (v_{r,n}(0^-))^{-1.5}$, in accordance with some experimental results in [174]. Hunt and Crossley [216] find that the law $e = 1 - av_{r,n}(0^-)$ is correct for collisions between nonlinear viscoelastic bodies with high approach velocities.

- The shapes of the bodies [173] [29] [468] [456]. For instance two spheres may possess a certain restitution. Now a rigid block striking a rigid horizontal surface, both made of the same material as the spheres, will exhibit in general a completely different behaviour. In particular Goldsmith [173] studied analytically the effect of the shape on the energy transformed into vibrations, and found that the minimum value occurs for spheres (which partially explains why experimental results are in closer accordance with theoretical ones for spheres colliding).

  - Their sizes [196] [11] [173].
  - Their masses [196] [11] [173].
  - Their elastic modulii [196] [11] [173].
  - The density of the medium in which they collide [128] [456].

All these different effects may be known to many. But they are worth recalling, since it is also true that it is often accepted that the restitution coefficients (Newton's as well as Poisson's) are material dependent: this is clearly only very partially true. See also remark 4.14 below.

Tatara [535] shows experimentally that when an external force acts on the considered system at the impact time, then the period of the shock and the restitution coefficient are different than when no external force is present. In fact, it may be considered that all shock process restitution coefficients are to be considered as "process constants", i.e. for a given process they take a certain value, that may be modified when one of these values changes. For example they may be considered
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exactly as the reduced damping $\zeta$ and natural frequency $\omega$ coefficients in a second-order equation $\ddot{x} + 2\zeta \omega \dot{x} + \omega^2 x = 0$: for a particular process, they are "constant", but they are in fact functions of the process parameters mass, damping and stiffness [59].

Remark 4.6 It seems that Newton in his Principia was aware of the fact that in more general cases than particles or spheres colliding, only normal components should be used to define the restitution [88]. But he wrote that $\epsilon$ should be independent of the momenta of the colliding bodies, which is erroneous [34].

4.2.1 Oblique percussions with friction between two bodies

The case of oblique impacts with friction is much more involved. By oblique it is meant that at the percussion instant, the centers of gravity of each body and the contact point are not aligned on the normal to the common tangent plane. Thus inertial effects play a role during the impact. Pioneering works on this topic may be found in [120] [430] [130] [461], see also [429] chapter 10 for a thorough treatment of shocks with or without friction, in 2 and 3 dimensional cases, using graphical tools. It has been the topic of an active research area during the last years, after Brach noticed that unrealistic solutions may occur with an improper treatment of the tangential impulse [55] and Kane's example [260] [261] on energetical inconsistencies in some impact problems based on Whittaker's method [583] when there is slip reversal at the impact, i.e. the final and initial relative tangential velocity have opposite signs; see for instance the works in [36] [37] [42] [55] [56] [57] [58] [274] [328] [345] [502] [519] [520] [522] [523] [570] [572]. A detailed analysis of the tip collision of a slender rod is made in [54] §6.2. In particular it is shown that for certain initial conditions $v_{i,n}(t_k^+), v_{i,t_1}(t_k^-)$ and certain values of the problem parameters (the problem is analyzed in the plane, so that $v_{i,t_2} \equiv 0$), then $\text{sgn}(v_{i,t_1}(t_k^-)) = -\text{sgn}(v_{i,t_1}(t_k^+))$. In the 3-dimensional case, the problem is even more intricate, since the sliding velocity may stop, and reverse in a direction not necessarily equal to that it had in the first phase.

Specifically, an impact problem consists mainly in determining post-collisions velocities. To this aim, relationships like Newton's rule or Poisson's rule [278] (that states that the ratio of the normal impulse during the compression phase $P_{n,c}$ and during the expansion phase $P_{n,e}$ is a constant in $[0, 1]$ sometimes denoted $\epsilon$ [572], $\epsilon_F$ [36], $R$ [57], $c$ [502]) may be used to calculate these unknowns, together with

13A constant fact in the "modern" literature on this topic is that almost every author claims he betters and corrects the preceding works. When this has stopped, perhaps a coherent and general impact dynamics theory will be settled!

14Notice that from its definition, Poisson's rule does not apply to rigid body dynamics, but only to compliant bodies. However as we saw in chapter 2, remark 2.1, an extension of compression and extension phases is possible to respectively pre and postimpact values of the momentum, being the impulse the sum of both. On the contrary, Newton's rule does not a priori precludes compliance.
some dry friction law. Kane’s example proves that these relationships known as Whittaker’s method [583] can yield a positive gain of kinetic energy at impact in case of oblique impact with friction, hence the necessity to develop new macroscopic models and introduce new coefficients. Among the aforementioned works, we may distinguish three main classes: i) Those that rely on a rigid body formulation, and thus lead to an algebraic formulation of the shocks dynamics, ii) Those that consider a non-zero shock duration, hence a differential analysis, iii) Those that more or less merge both in the analysis.

A common feature of these studies is that they aim at providing a reasonably general approach to the solution of impact bodies: engineers need methods for solving percussion problems, and proper treatment of experimental impact data requires an organized approach including a good model.

4.2.2 Rigid body formulation: Brach’s method

In these works the classical algebraic equations that relate velocity jumps and impulses at the percussion instants are used to derive the expression for the loss of kinetic energy $T_L$ at impact. One of the main advantages of this formulation is the simplicity of the obtained equations and of the subsequent analysis, in comparison with integration of differential equations. Basically in [54] [55] [56] [58], Brach deal with the planar and three-dimensional cases, and derives $T_L$ as a nonlinear (second order polynomial) function of Newton’s coefficient $e$, and of an equivalent coefficient of friction or impulse ratio $\mu \triangleq \frac{P_t}{P_n}$, where $P_t$ and $P_n$ are the tangential and normal impulses respectively at the contact point, i.e. $P_n = p_{i,n}$ and $P_t = \sqrt{p_{i,t1}^2 + p_{i,t2}^2}$.

For the case of two particles colliding with friction, it is found that the kinetic energy loss is given by [55] [174]

$$T_L = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (v_{r,n}(t_k^-))^2 \left(1 + e\right) \left((e - 1) + 2\mu r + (1 + e)e^2\right)$$

(4.42)

where $r = \frac{v_{i,t1}(t_k^-) - v_{i,t2}(t_k^-)}{v_{r,n}(t_k^-)}$. It is supposed that $v_{i,t2}(t_k^-) = 0$, i.e. the initial tangential velocity is along $t_{11}$.

Remark 4.7 Péres [429] proposed the following rule for impacts with dry friction (15):

Both coefficients may not be always equal, see subsection 4.2.4.

15Whittaker [583] states that if $v_i(t_k^+) = 0$ then the percussion vector $P \in \text{Int}(C)$, whereas if $v_i(t_k^-) \neq 0$ then $P \in \partial C$. Kane and Levinson [261] use a similar idea but use a static and a dynamic friction coefficients. Smith [503] proposes another rule to relate tangential and normal components of the percussion vector, see subsection 4.2.3. The most established way of treating those phenomenons is the one by Moreau [381], that we describe in example 4.1 and in subsection 5.4.2.
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- If there is a unidirectional sliding velocity during the collision, then \( \frac{P_t}{P_n} = \mu \). In other words, the percussion vector lies on the boundary of the friction cone, \( \partial C \).

- If there is a change of the direction of sliding (or tangential velocity reversal), then \( P_t \leq \mu P_n \) (i.e., the percussion lies inside the friction cone, i.e., in \( \text{Int}(C) \)).

Pérès’ approach treats the coefficient \( \mu \) similarly as a Coulomb’s friction coefficient, and is therefore different from that in [54], according to whom \( \mu \) is the impulse ratio and is to be determined so that it yields \( T_L(e, \mu) \leq 0 \). In fact more exactly, in a practical impact problem, one can use this criteria to check whether the set of coefficients associated \( a \text{ priori} \) or after experiments to the bodies is consistent [54] §5.4: if it is not, it may signify that either the coefficient values are wrong, or that the rigidity hypothesis is not justified; but the works in [55] [56] do not propose any method to \( a \text{ priori} \) compute the value of the coefficient for a given problem [57]. Bounds on the impulse ratio are rather derived, that guarantees energetic consistency of the model: they are called critical values [54]. The impulse ratio is related to the coefficient of friction, but also depends on the inertial properties of the projectile, and dynamic effects of strain rate. It is easily seen that since the impulse is the integral of the force, the ratio between the tangential and the normal parts of the impulse may differ from that between the tangential and normal forces: consider for instance the case when during the shock process, the tangential relative velocity reverses or simply is zero, so that the normal force may be anywhere inside the friction cone. Then the impulse represents a sort of average of the force so that the impulse ratio is a sort of average of the normal and tangential forces ratios. This does not prevent from defining Coulomb’s like laws for the impulse ratio, as proposed for instance by Pérès [429] and Moreau [381]. This topic is thoroughly covered e.g. in [54] §6.3.1, where many different experimental data are analyzed. Experimental results for the determination of the impulse ratio can also be found in [558] (impacts of steel spheres on plane surfaces made of various alloys; zinc, armco-iron, steel, high-alloyed steel of the austenitic type), [524] (impacts of hard particles with plastic and rubber specimens). Dynamic effects of strain rate are however not taken into account. Note that all the effects that play a role in the determination of \( \mu \) are specific to the collision process, where in particular high pressure may exist at the contact point. Hence the friction models for shock processes may be quite different from those used when only sliding occurs, see e.g. [15].

Let us first be more precise with sliding: if we still rely on the perfect rigidity assumption then as we have seen the impact duration is zero. Hence there cannot be any “sliding” of the two bodies, but more exactly a jump can occur in the tangential velocity at impact. For instance, consider equation (4.25). First notice that even if there is no friction, inertial coupling due to the fact that the matrix \( M_A \) is not necessarily diagonal may imply discontinuities in all the components of \( V_{A_i} = \begin{pmatrix} v_{i,n} \\ v_{i,t_1} \\ v_{i,t_2} \end{pmatrix} \) expressed in the local frame \( (n_1, t_{11}, t_{12}) \) at the contact point.
instance a slender rod striking a horizontal rigid surface with zero initial tangential velocity (i.e. \( v_{i,t} (t_k) = v_{i,t_0} (t_k) = 0 \)), can undergo a jump in the tangential velocity [54] §6.2.1. For an explicit calculation of such an example, see subsection 6.3.1 where the dynamics of a lamina striking a rigid surface are derived. Now in the simplest case of a particle striking a plane, there are only two degrees of freedom and equation (4.25) reduces to \( ma_{v_n} (t_k) = \mu p_n, \) \( ma_{v_t} (t_k) = \# p_n. \) Hence friction or tangential compliance only may imply a jump in the tangential velocity.

In the planar case [55], the value of \( T_L \) that corresponds to the sliding case \( T_{L, s} \) is used, i.e. when \( \tau \neq 0 \) in (4.42). Then it is argued that for \( \mu < \mu_m \) (where \( \mu_m \) maximizes \( T_L \) for fixed \( \epsilon \)) there is sliding when the bodies separate, and that \( \mu \geq \mu_m \) implies equal final tangential velocities. The reasoning in [55] is the following: for \( \mu < \mu_m \), the sliding hypothesis works; for \( \mu = \mu_m \), the loss of energy is maximum and this is the first case when final velocities are equal, since an amount of friction tends to "slow down" the bodies; now for any additional amount of friction, the bodies stick before the end of the impact, hence it is incorrect to use \( T_{L, s} \) with \( \mu > \mu_m \); this last point is supported in [55] by the fact that there cannot be velocity reversals due to passivity of the frictional model, a statement that is true for central collisions (hence in particular point mass or particles collisions), as pointed out in [56] and [519], see also [429] chapter 10 §21. This work is generalized in [56] to the three-dimensional case. Of particular interest is the slender rod tip impact problem, where the kinematical equations show that tangential velocity reversal occurs for small enough initial value (In general the slip process depends on the values of the coefficients, the initial velocities and the kinematics [519]). It is assumed in [56] that \( 0 \leq \epsilon \leq 1 \) but as pointed out in [503], it is not evident why this should be true in general contrarily to the frictionless or central impact cases where \( T_L \) implies it.\(^\text{16}\) In fact as we shall see further Brach’s method is rather useful to determine lower and upperbounds on restitution coefficients that yield a consistent impact process. The method is extended in [54] to finite contact areas, with an additional moment coefficient of restitution that seems necessary to solve the impact problem (see also [55]). A very interesting point in [54] is the consideration of a tangential coefficient \( e_t \) of restitution to account for tangential compliance when a hard object strikes a compliant surface, see [54] p.30 and pp. 132-134. The need for considering such tangential compliance is pointed out for instance in [519] [503], and has been experimentally evidenced in [352] [353] [156], as well as in [243] [244] using sophisticated theories of elastic impacts. From our above discussion on "rigid" slippage (i.e. jumps in the tangential velocity), it seems quite natural to consider such a coefficient. This is introduced in [54] equation (2.28) exactly as Newton’s

\(^{16}\)Such conclusion clearly dissociates oblique percussions with friction from frictionless or central impacts. Similarly, it has been shown [56] [187] that in some cases, \( T_L \) increases as \( \epsilon \) increases for certain friction coefficient values. It is also pointed out in [502] an experiment that consists of a superball (a kind of ball made of rubber and that possesses a high restitution coefficient when colliding almost any rigid material, so that it rebounds very high when dropped on the ground) bonded at the end of a slender rod. When collision occurs against a rigid surface, measurements provided values of Newton’s coefficient \( e \in [0.7, 1.4] \) depending on the initial orientation of the rod and on friction.
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rule, replacing normal velocities by tangential ones. In summary, discontinuities in the tangential velocity may arise from different sources:

- Inertial effects (even for frictionless constraints)
- Effects related to dry Coulomb's friction
- Effects related to tangential compliance, that may be modelled through a tangential restitution.

Note that tangential compliance of the bodies surfaces that contact may not play any role if there is no dry friction.

Further generalizations for the three-dimensional case are presented in [58], not a priori assuming $e \leq 1$ and incorporating some results in [519]. To conclude this part, let us summarize Brach's approach [59] [58]:

a) Use linear and angular momentum theorems (see (4.25) and (4.36)) for rigid bodies percussions, that yield algebraic linear relationships between pre and post impact velocities

b) Use kinematic and kinetic restitution rules to complete the set of equations so that there are as many equations as unknowns

c) These equations provide postimpact velocities and $T_L$ in terms of initial ones, inertial properties, initial conditions and coefficients, and can be used to develop bounds on the coefficients using kinematic constraints and/or work and energy conditions

d) The post impact velocities and the coefficients bounds are not restricted to point contact (recall there may be rigid surface contacts), are independent of the specific nature of the contact processes, unless a contact process condition is used to establish one of the bounds

e) Specific contact process models can be used to relate the above general expressions (the equations, $T_L$ and coefficients bounds) to the physical process and analytically, numerically or experimentally\footnote{See e.g. [224] for an experimental evaluation of $T_L$} evaluate the coefficients. For instance Hertzian, vibrations or finite element theories [243] [244] [614] can be used to relate $T_L$ to the dimensions, elastic Young modulus ... and then deduce the coefficients from them.

Let us reiterate that rigorously one should study and use these compliant models or theories "in the limit" to incorporate them in the rigid analysis case. An "engineering" point of view is to investigate whether the method generally provides accurate
prediction in experiments, without any further considerations. Concerning step a),
let us assume that the percussion vector in (4.25) is of the form

$$P_i = \begin{bmatrix} p_{i1} \\ p_{i2} \\ p_{i3} \\ p_{i,n} \\ p_{i,t_1} \\ p_{i,t_2} \end{bmatrix}$$

(4.43)

Then from (4.25) it follows that in order to be able to calculate the postimpact
velocities and all the components of $P_i$ (i.e. $12 + 6 = 18$ unknowns to the problem),
one must add some relationships between known values and unknown ones. One
solution that comes naturally to one’s spirit is to extend Newton’s conjecture to all
the velocities, i.e. to set

$$\omega_{11}(t_k^+) - \omega_{12}(t_k^-) = -e_{\omega,1}(\omega_{11}(t_k^-) - \omega_{12}(t_k^-))$$
$$\omega_{21}(t_k^+) - \omega_{22}(t_k^-) = -e_{\omega,2}(\omega_{21}(t_k^-) - \omega_{22}(t_k^-))$$
$$\omega_{31}(t_k^+) - \omega_{32}(t_k^-) = -e_{\omega,3}(\omega_{31}(t_k^-) - \omega_{32}(t_k^-))$$
$$v_{r,n}(t_k^+) = -e_n v_{r,n}(t_k^-)$$
$$v_{1,t_1}(t_k^+) - v_{2,t_1}(t_k^-) = e_{t,1}(v_{1,t_1}(t_k^-) - v_{2,t_1}(t_k^-))$$
$$v_{1,t_2}(t_k^+) - v_{2,t_2}(t_k^-) = e_{t,2}(v_{1,t_2}(t_k^-) - v_{2,t_2}(t_k^-))$$

(4.44)

which is step b). For instance torsional restitution is introduced in [54] §6.5 (see also
[10]), motivated by experimental results in [211]. Hence there are $6 + 12$ unknowns
(percussion and postimpact velocities), and $12 + 6$ equations (dynamics and restitu-
tion laws). Notice that the first three (kinematic) coefficients in (4.44) are related
but not equal to the moment coefficients proposed in [54] (see the paragraph below
on equivalence of coefficients). Also the $e_{\omega,i}$’s above do not have to be positive,
allowing for nonreversal of the angular velocity.

4.2.3 Additional comments and studies

Smith [503] derives $T_L(t_k)$ as a quadratic function of the relative twist between
the two colliding bodies. This expression allows to derive a generalization of $T_L$ in
(4.42). The associated ellipsoid is used to visualize the different outcomes of the
impact problem. In particular the maximum value of $T_L$ is shown to occur for a
relative velocity $v_r(t_k^+) = 0$, corroborating the reasoning in [55] [56]. Smith [503]
introduces a new definition of $\mu$ that involves an "average" of the final and initial
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values of $v_{r,t} = (v_{1,t1} - v_{2,t1})t_{11} + (v_{1,t2} - v_{2,t2})t_{12}$, i.e.

$$
\mu \frac{|v_{r,t}(t_{k}^-)||v_{r,t}(t_{k}^0)| + |v_{r,t}(t_{k}^0)||v_{r}(t_{k}^0)|}{|v_{r}(t_{k}^0)|^2 + |v_{r}(t_{k}^0)|^2}
$$

(4.45)

It is then shown that the found $T_L$ is always $\leq$ for Newton’s coefficient $e \leq 1$ and with the impulse ratio as in (4.45). It is shown on Kane’s example for a particular fixed value of $\mu$ that the new definition predicts loss of energy whereas the other one predicts a gain. Note that the basic idea in [261] is to choose a definition of $\mu$ such that its sign is that of the final value of $v_{r,t}$, whereas the new definition in [503] does not neglect what happens initially. In [56] the sign of $\mu$ is chosen as the sign of the calculated value $\mu_{\text{max}}$ that maximizes $T_L$ when $e = 0$, who also pointed out the problem related with dependence of $\mu$ on relative velocity. Note that both philosophies in [55] [56] [58] and [503] are different since the first one examines the possible sets of values of $e$ and $\mu$ that yield a consistent solution and can be used in the actual solution ($\mu$ being nevertheless by assumption upperbounded by some "critical" values), whereas the second one proposes a more or less ad hoc expression of $\mu$ that is shown to be always consistent, and thus should fit within the former’s framework. Most importantly note that the choice of the impulse ratio in (4.45) is shown in [503] to fit quite well (for the case of two spheres colliding) with the experimental results in [351].

Equivalence of coefficients

In the general three-dimensional rigid problem, it appears that one restitution coefficient is needed for each unknown contact percussion component [59]: indeed, each time one supposes that the percussion vector possesses a certain form, then one associates at the same time (implicitly) a restitution coefficient that is needed to solve the impact problem. This is clear looking at (4.25). As we shall see in chapter 6, the definition of such coefficients is not an arbitrary way to treat rigid body impacts. They are actually needed to solve the problem, may they be constant or not. In a tangential coordinate direction, one may choose equivalently a kinetic coefficient $\mu$ (that may be positive or negative to control velocity sign changes) or a kinematic one $e_t$ [54] p.30. It may be shown [54] §2, that $\mu$ and $e_t$ can be related (recall velocities and impulses magnitude are algebraically related, see (4.25) and (4.44)), although they represent different physical effects, so that in fact the choice of either one coefficient or another is a matter of convenience [57] in the rigid bodies problems case. In particular $e_t$ may take negative values that represent the fact that there may be slip reversal or not (the relative slippage velocity sign changes). Notice that a negative normal restitution is forbidden (this would imply penetration), but in case of tangential motion such negative values must be permitted to allow non-reversal consideration. Mimicking the normal process, one may ask which kind of unilateral constraint has to be considered in this case to model such effects, and if it is reasonable to adopt this point of view. It might be that a positive tangential restitution corresponds to a unilateral constraint (and
hence the coefficient of restitution can be defined as a constant of the materials like Newton's coefficient), whereas a negative one has no such direct interpretation: the phenomenon stems from other physical data like bodies' inertia, shape, position..., and it may be better then to assume $e_t$ is not a constant, but a function of these data and of other coefficients [54] equation (2.31). For instance [322] [568] consider Coulomb friction and choose to fix $e_t = e_{t,0} \in [0, 1]$ when there is no sliding, whereas $e_t = e_t(f, e, \text{relative velocity, inertia})$ when there is sliding, where $f$ is the Coulomb friction coefficient. These authors thus a priori associate tangential restitution to friction, which apparently is not always the case elsewhere [54] [519]. Note however that [54] equation (6.24) and [322] equation (5), as well as [54] equation (6.25) and [322] equations (6)-(7) are the same. Since velocity reversal may be due also to friction and inertial conditions, both effects can play a role together: as always this will depend on the considered problem. In fact, tangential contact processes can be complicated, and without friction, tangential forces might not exist and a tangential compliance would not receive any energy [59]. In some cases, when the contact process is better known, it may be better to tailor the model (see e.g. [54] chapter 6, figures 6.23 and 6.24 for a "bilinear" model). In some other cases (spheres colliding), tangential reversals are apparently always due to tangential compliance, and not to Coulomb friction [523]. The concept of equivalence between coefficients is important to relate the results of authors that may or may not use the same coefficients to lead their calculations.

4.2.4 Differential analysis: Stronge's energetical coefficient

We now deal with analytical studies based on the assumption that the bodies undergo a deformation at the impact point $A$, but that this deformation is local only (i.e. it is negligible at the bodies' scale). Such assumptions were for instance done by Péres and Delassus [123] [430] [429] chapter 10, section 5, following studies by Darboux [120]. For the sake of brevity and because it is the most representative work in this area, we describe Stronge's work in [519] [520] in some detail. Notice that such analysis may or may not rely on an explicit model of the bodies at contact. Some works (see [243] [244] and references therein) strongly rely on particular sophisticated models to deduce the system's motion and energetical behaviour, coefficients of restitution ... Another class of analysis adopts a "macroscopic" viewpoint and assumes no particular contact model. This is what we describe now. A common assumption in this class of studies is that the positions remain constant during the shock, which allows to simplify the dynamics to similar expressions as in the rigid case (see e.g. [36] equation (5), [519] equation (6), and see also [243] [244]), by considering the impulse $P(t_f) = \int_{t_0}^{t_f} F(\tau)d\tau$ over the shock interval.

Stronge [519] [520] studies the 2-dimensional dynamics of a lamina striking a static plane (i.e. body 1 is fixed and plays the role of a constraint, so that $v_{r,n} = -v_{2,n}$ with the notations chosen above). Tangential compliance is neglected and
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Coulomb's friction rule is adopted. The main goal of the work is in fact to prove that the inconsistency of some impact problems comes from neglecting dependence of \( e \) on the slip process \([520]\). The impact process is divided in compression \([0, t_c]\) with \( v_{2,n}(t_c) = 0 \), and expansion phases \([t_c, t_f]\). Unidirectional slip is assumed on \([0, t_c]\). \([519]\) introduces the concept of characteristic normal impulse \( p_{n,j} = m_j v_{2,n}(0) \), \( j = \text{sign}(v(t(0))) \), and \( m_j \) is an equivalent normal mass at the contact point \( CP \) (\( m_j \) is a function of the mass, radius of gyration and \( CP \) coordinates). The subscript \( j = \pm 1 \) is introduced to encompass the both cases of positive or negative tangential approach velocities in a single notation. Note that \( p_{n,j} \) corresponds to the normal impulse that terminates the compression phase. Indeed the dynamics of the system leads to \( v_{2,n}(t) = v_{2,n}(0) - \frac{p_{n,j}(t)}{m_j} \) and noting that \( v_{2,n}(t_c) = 0 \) the result follows. It is shown that velocities variations on \([0, t_c]\) are proportional to the relative impulse \( \frac{p_{n}(t)}{p_{n,j}} \), where \( p_n(t) \) is the normal impulse at time \( t \) (Indeed from the above one finds that \( \frac{v_{2,n}(t)}{v_{2,n}(0)} = 1 - \frac{p_n(t)}{p_{n,j}} \)). Then two coefficients \( \gamma \) and \( \tau \) are introduced that quantify the part \( \gamma p_{n,j} \) of \( p_{n,j} \) that stops slip on \([0, t_c]\) \([18]\) and the rest of the characteristic impulse \( (\tau - \gamma)p_{n,-j} \) on \([t_c, t_f]\). If \( 0 \leq \gamma \leq 1 \) slip stops on \([0, t_c]\); if \( 1 < \gamma \) and \( \tau > \gamma \) slip stops during expansion. If the tangential velocity reverses during the shock process this yields \( \frac{v_{2,n}(t)}{v_{2,n}(0)} = 1 - \tau \) and \( \frac{v(t)}{v_{2,n}(0)} = \frac{(\tau - \gamma)\mu_{m-s}}{n_j} \), where \( n_j \) is the tangential equivalent mass at \( CP \). Both reversal and stick cases are encompassed. Then Stronge \([518]\) calculates the work \( W_n \) and \( W_t \) performed by the normal and tangential forces in term of \( p_j, \gamma, \tau, v(t(0)), v(t(t)), v_{2,n}(0), v_n(t(t)) \):

\[
W_n = \frac{1}{2} p_{n,j} v_{2,n}(0)(2 - \gamma) + \frac{1}{2} p_{n,-j} v_{2,n}(0)[(1 - \gamma)^2 - (\tau - 1)^2]
\]

\[
W_t = \frac{1}{2} j \mu p_{n,j} v(t(0)) \gamma - \frac{1}{2} j \mu p_{n,-j} v(t(t)) (\tau - \gamma)
\]

where \( T_L = W_n + W_t \) (recall that in the perfectly rigid case, only the kinetic energy loss has a meaning, see chapters 1 and 2).

The rest of the study is devoted to compare three definitions of the coefficient of restitution: Newton's rule, Poisson's rule, and a new energy coefficient \( e_* \) in \([520]\) \( E \) in \([57]\) \([19]\), and that we can name Stronge's rule:

\[
e_*^2 = \frac{\text{elastic energy released on } [t_c, t_f]}{\text{elastic energy absorbed on } [0, t_c]}
\]

\( e_* \) is found \([519]\) to be \( e_*^2 = \frac{W_{n,e}}{W_{n,c}} \) when there is no tangential compliance, where \( W_{n,e} \) and \( W_{n,c} \) are the works performed by the normal impulse during expansion and compression phases respectively. Such result is natural since in absence of tangential compliance, the elastic effects are normal only. Another way to express Stronge's coefficient is

\[
\int_{t(t_1)}^{p_n(t_f)} v_{2,n}(p_n)dp_n = -e_*^2 \int_0^{p_n(t_1)} v_{2,n}(p_n)dp_n(t)
\]

\( 18 \) \( \gamma p_{n,j} \) can be calculated from the equations of motion \([523]\).

\( 19 \) Stronge's coefficient extends an expression of the restitution given by Routh for smooth bodies \([461]\) and by \([53]\), see also \([429]\) chapter 10 §24.
where we have used the fact that the impulsion of the normal force is given by
\[ p_n(t) = \int_0^t F_n(t) dt, \]
and hence \( dp_n = F_n dt \). Thus
\[ F_n(t) v_{2,n}(t) dt = v_{2,n}(p_n) dp_n. \]
This is a valid time rescaling \(^{20}\) because \( p_n(t) \) is a monotonic function and \( p_n(0) = 0 \). Hence there exists a strictly increasing function \( f(t) \) such that \( p_n = f(t) \), and the inverse function exists. Consequently \( v_{2,n}(t) \) can be expressed as \( v_{2,n} f^{-1}(p_n) \equiv v_{2,n}(p_n) \) shortly. More details on the use of this time-scaling are given in subsection 4.2.6.

**Remark 4.8** As we have pointed out in remark 2.1, in some cases it is possible to define the limit of compression and expansion phases as pre and postimpact times. However we have also seen that the distribution theory does not allow to give a meaning to the work at impact times, because it involves the product of a discontinuous function with a Dirac measure. Hence it seems that Stronge coefficient could be given a meaning in the rigid case by replacing the work of the forces by the kinetic energy. One may split the kinetic energy into two terms, one for normal velocities and the second for tangential ones. Then Stronge’s coefficient is a constraint on the normal kinetic energy loss. Following this reasoning, let us note that Stronge’s energetic coefficient may be seen as a work-energy constraint saying that \( W_{n,e} = e^2_n W_{n,c} \).

The role of friction is not really clear in these developments since \( W_n \) is likely to depend on friction when tangential and normal directions are dynamically coupled, although it is claimed in [520] that \( e_n \) is independent on friction. This point of view has in fact historical roots according to which the normal process is independent of the frictional-tangential effects (This reasoning fails if tangential compliance exists, since in this case the tangential process is no longer a consequence of the normal one).

Based on the foregoing developments, each value of the restitution coefficient (Newton, Poisson and energetic) is calculated as a function of \( \tau, \gamma \) and \( P_j \), depending on whether there is slip reversal or not, and the works \( W_n \) and \( W_t \) are computed in each case. For instance, Newton’s coefficient is calculated as \( e_n = \frac{v_{2,n}(t_f)}{v_{2,n}(t_0)} \) and is found to be \( e_n = \tau - 1 \). Poisson’s coefficient \( e_p = \frac{P(t_f)-P(t_e)}{P(t_e)} \) has a value which depends on the slip process: for example when there is slip reversal and \( \gamma < 1 \), one finds \( e_p = \frac{\tau-1}{(1-\gamma)+\tau P_{n,-1}} \). When slip is unidirectional then \( e_p = \tau - 1 \) so that in this case Newton’s and Poisson’s conjectures are equivalent \(^{21}\). However when slip stops or reverses then \( e_p \neq e_n \). Finally the same operation is done to compute \( e_n \).

---

\(^{20}\) Consider an ODE \( \dot{z} = f(x,t) \). Assume that \( \tau = g(t) \) for some strictly increasing function \( g(\cdot) \). Hence \( g^{-1}(\cdot) \) exists and is strictly increasing as well. Simple calculations then allows to write the ODE as \( \frac{dx}{dt} = f(x(t),\tau) \) with \( f(x(\tau),\tau) = \left( \left[ \frac{dx}{dt} \right] \right)^{-1} f(xg^{-1}(\tau),g^{-1}(\tau)) \). Such manipulations can be sometimes useful in the study of ODE’s, see e.g. [469] [408].

\(^{21}\) This is what is meant in the related literature when some authors state that under certain conditions these two coefficients are equal.
\[ \gamma < 1 \text{ and slip reversal, one finds } e_* = \frac{(\gamma-1)^2}{(1-\gamma)^2+\gamma(2-\gamma)\rho_{n-2}}. \]  For unidirectional slip, then \( e_* = \tau - 1 \), which confirm that all three coefficients are the same when there is no tangential velocity reversal.

Stronge’s rule is shown to be the only one that is always energetically consistent: Newton’s rule can result in \( T_L > 0 \) when slip reverses, Poisson’s rule always dissipates energy (this is consistent with the conclusions in [234]) but is not satisfactory since non-frictional dissipation does not vanish when the coefficient equals 1. All three rules are equivalent if slip does not reverse nor stops. [519] also provides 4 kinematic conditions that identify the different types of slip process, and deduces from them 2 parameters \( \mu_1 \) and \( \mu_2 \) depending on \( CP \) coordinates and the radius of gyration \( k \). Finally depending on friction, \( CP \) coordinates, \( k \), \( \mu_1 \), \( \mu_2 \) and initial velocities signs, it is possible to identify the slip process during the collision. [519] suggests that \( e_* \) is the coefficient of restitution, essentially material dependent. According to [519], a strong property of \( e_* \) is that it is by definition bounded between 0 and 1, for any impact process, a property not shared by the other two coefficients for which simple numerical bounds do not always exist (\( \varepsilon \) and Poisson’s coefficient generally depend on the tangential impulse [57]). The results in [519] are confirmed in [234] using a different way to calculate \( T_L \).

In summary, the basic idea is to calculate the various coefficients of restitution from their definition, and using the system’s dynamics during the shock (which are derived doing the assumption of constant position during the collision). Then introduce the computed values into the kinetic energy loss at impact, and investigate the cases when \( T_L \leq 0 \). This allows to find whether a coefficient may or may not yield energetical inconsistency. The reader is referred to [520] for a clear exposition of Stronge’s ideas.

**Remark 4.9** One drawback of the energetical coefficient is that the linearity of the relationship between post and preimpact values is lost.

**Example 4.1 (Sphere against a rigid wall)** Further work in [523] is based on the above time-scaling of the relative velocity with respect to the normal impulse \( P_n \) to identify the slip process during impact, see subsection 4.2.6 and (4.63) for further details. This is based on using \( e_* \) and Coulomb’s friction. This allows to retrieve known results (see e.g. [429] §7,b), like the fact that the direction of slip of the contact point for two spheres colliding, is constant during any percussion (This was proved in the 19th century by Coriolis [113] and Phillips [436]). To get more insight on this example, let us consider the planar case of a sphere colliding a rigid barrier. Let us denote for simplicity \( n = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( t = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) the normal and tangential unit vectors at the contact point \( A \). The normal and tangential velocities are denoted as \( v_n \) and \( v_t \), and the angular velocity as \( \omega \). Then the shock dynamical

\[ ^{22}\text{Note that a similar analysis is done in [429] chapter 10 §20.} \]
CHAPTER 4. TWO BODIES COLLIDING

The equations are

\[ m \sigma_{v_n}(t_k) = p_n \]
\[ m \sigma_{v_t}(t_k) = \frac{7}{3} p_t \]  \hspace{1cm} (4.49)
\[ \frac{2}{5} m v^2 \sigma_{\omega}(t_k) = - r p_t \]

Together with the restitution rule \( v_n(t^+_k) = -e v_n(t^-_k) \), and the fact that the percussion \( P = \left( \begin{array}{c} p_n \\ p_t \end{array} \right) \) belongs to the friction cone \( C \). By doing the assumption that the collision process is not instantaneous (differential-like analysis) and using the normal impulse time-scaling, it is possible to show [523] that \( \frac{d v_n}{d p_n} \) is negative (a similar procedure is employed in [429]). Hence one deduces that the velocity \( v_t \) varies monotonically. From the dynamical equations in (4.49), one deduces that

\[ v_t(t^+_k) = v_t(t^-_k) + \frac{7}{2} \mu v_n(t^-_k) \]  \hspace{1cm} (4.50)

It is then easy to see that if for instance \( v_t(t^-_k) > 0 \) (in which case \( p_t < 0 \) and the percussion lies on the friction cone), then \( v_t(t^+_k) = 0 \) for \( \mu = \mu_0 = \frac{7}{4} \frac{v_t(t^-_k)}{(1+e)v_n(t^-_k)} \) (see [429] [523]). It is argued e.g. in [523] [429] that if \( v_t \) attains zero, then sticking occurs, because then there is no tangential force that may influence its variation anymore.

If one deals with instantaneous shocks, it is not clear how this kind of arguments may be applied. The only thing that one can use are the dynamical equations in (4.49), the restitution rule, and \( P \in C \). Let us apply Moreau’s rule [381], that we shall retrieve in subsection 5.4.2 when we deal with tangential impacts (see (5.100) and (5.101)). Note that it corresponds to soft shocks, i.e. \( e = 0 \). It is shown that according to this rule, there is always a unique solution in terms of postimpact velocity and percussion vector. The percussion vector is defined as the closest point in \( C \) to the point

\[ \begin{pmatrix} m & 0 \\ 0 & \frac{7}{2} m \end{pmatrix} \begin{pmatrix} v_n(t^-_k) \\ v_t(t^-_k) \end{pmatrix} \]  \hspace{1cm} (5.100)

If it happens that \( P \in \text{Int}(C) \), then \( v_t(t^+_k) = 0 \) since \( \sigma \) in (5.100) (5.101) is zero (as well as \( v_n(t^+_k) \) since anyway \( e = 0 \)). If on the contrary \( P \in \partial C \), then \( v_t(t^+_k) \neq 0 \). This provides the value of \( P \), from which one can compute that of \( v_t(t^+_k) \) through the computation of \( \sigma \), see (5.100) (5.101), as \( \sigma = -\frac{2}{3} m p_t + v_t(t^-_k) \). Since for \( v_t(t^-_k) > 0 \) one has \( p_t < 0 \), it follows that \( \sigma = -\frac{2}{3} \mu v_n(t^-_k) + v_t(t^-_k) > 0 \). Finally \( v_t(t^+_k) = \sigma \) shows that Moreau’s rule in (5.100) (5.101) yields that there is no tangential velocity reversal for the planar sphere colliding a wall with friction. Such a result is in a sense reassuring, because it shows a coherence between various studies, based on different assumptions and formulations. However, the relationship between both requires more care.

A detailed analysis of the dynamics of two rigid spheres colliding with friction is also provided in [503], using a particular impulse ratio guaranteeing \( T_L \leq 0 \) for all \( e \leq 1 \) (see (4.45)). Batlle [36] treats the bounding of a sphere on a fixed surface
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and finds that Newton’s or Poisson’s rules and Coulomb’s dry friction always yield $T_L \leq 0$.

4.2.5 Additional comments and studies

In [36], Batlle derives general conditions for $n$-degree-of-freedom systems under which Newton’s and Poisson’s rules are consistent and equivalent, using the expression of $W_n$ from [536] and studying sufficient conditions when these conditions are true (hence extending the study in [519]). Keller [274] proposes to use Poisson’s and Newton’s rules to derive a general method extending analytically the graphical Routh’s procedure that allows to investigate whether the relative velocity $v_{r,t}$ reverses or not during impact. The result is that $\frac{\mu}{\mu} = f$ (Coulomb’s rule) when $v_{r,t}$ does not vanish, whereas $\frac{\mu}{\mu} < f$ otherwise, i.e. Whittaker’s assumption is true only when there is unidirectional nonzero slip at contact. Evidently the major drawback of the method is that it requires integration of ODE’s on the impact interval. The trick of the time rescaling of the dynamical equations during the collision in terms of normal impulse is used, see (4.63) and (4.66). The work is extended in [275] to include frictional moments. We shall come back on Keller’s shock equations in subsection 4.2.6. In [502] simplifying assumptions on the shock process are made to express the ratios between Newton’s $e$, Poisson’s $e_p$ and Stronge’s $e_s$ coefficients, which are shown to depend on friction, inertia and initial velocities, i.e. $e = e_s = f(\lambda, \mu, \theta)$, where $\mu$ is the Coulomb’s coefficient, $\theta$ is the rod initial orientation, and $\lambda$ is an inertial term. A comparison of the theoretical results obtained via restitution coefficients and via $e_s$ numerical simulation based on a finite elements method is made: this is an interesting point since it might be necessary to use sophisticated compliant models to study oblique impacts with friction with both tangential and normal compliances, and deduce new "rigid" rules as the limit of these. Also it seems quite logical to use as a validation for nonmodel-based studies contact-impact problems based on sophisticated models of the bodies. The works in [57] [243] [244] [502] are steps in this direction. The results in [502] show significant discrepancies between both methods. It is then concluded by the authors that a good macroscopic model should be capable of separating dissipation due to sliding (dry friction) and due to the normal and tangential deformations (i.e. in particular take into account tangential compliance). Therefore Brach’s approach that consists of defining a tangential restitution coefficient goes in the right direction.

Brach [57] proposes to compare theoretical predictions with rigid model and simulation results based on an approximation procedure (time-discretization together with algebraic equations relating initial and final velocities) of the impulse ratio $\mu$, for a compliant model. The benchmark example of a lamina striking a wall is chosen. Quite interestingly, some simulation results yield $e_s > 1$. We are tempted to relate this with the fact that in certain cases, increasing the kinematic normal coefficient (i.e. the normal velocity increases during impact) yields a decreasing $T_L$, because at the same time friction dissipates more energy [56] [187]. It might be
that Stronge's definition misses the friction dissipation by taking into account only normal quantities, see remark 6.26. Another possible explanation is that during the impact, some angular energy may be converted in linear normal energy [59]. This may happen when for instance a slender rod collides an elastic surface. If the rod's orientation changes during the shock, then the effective inertia at the contact point may be smaller during expansion than during compression. Hence there may be a positive acceleration in the normal direction. Note that this last explanation would imply angular position discontinuity in the rigid limit case, which is precluded by the theory developed here.

The comparative results in [57] show that there is agreement between the compliant and rigid body assumptions in most of the tested cases, although as recognized in [57] much more work is needed: in fact the problem attacked in [57] is that of studying the validity of a limit problem (rigid bodies) by comparing it with a compliant problem to see if both agree. This may be of great practical importance, because algebraic equations are much more tractable than differential ones: thus it would be nice to get rid of the latter ones by proving that in any case, the rigid body assumption provides accurate enough results. The mathematical foundations of such studies lie in study of convergence of smooth problems towards nonsmooth ones, that we have described in chapter 2. This is not a pure formal and elegant way of proceeding, since if it can be shown for instance that a sequence of smooth problems $P_n$ converges "strongly" (this word is to be understood here only from an intuitive point of view) towards a nonsmooth problem $P$, one can legitimately expect that the solutions of $P$ are "closed" (still intuitively) to those of $P_n$.

Multiple micro-collisions phenomenon

Closely related to the work in [57] is the recent investigation in [516]. Roughly, Stoianovici and Hurmuzlu [516] present experimental results of a slender rod falling on a rigid obstacle. Since most of the papers in the area consider this system as a benchmark example for various investigations, the results are of great interest. The authors show that the restitution coefficient varies with the orientation of the rod at the impact time, and hence question the validity of the rigid body assumption for their system. In particular they point out the importance of the multiple micro-collision phenomenon that occurs at the impact time: the impact in fact involves successive small collisions between the rod's tip and the obstacle (The phenomenon had also been noticed in [174] [596] and is due to the flexibility in the bodies). This phenomenon is not modeled if rigidity is assumed and if the period during which the micro-collisions occur is taken as an instant of time $t_k$ (23). This last

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23 It is noted worthy that this phenomenon also calls into question the assumption on which Poisson's coefficient is based, and the fact that Poisson's conjecture is better that Newton's one [278]: indeed it shows that the impact phenomenon may be much more complex than merely a compression and extension phases. Hence it could well be that assuming some flexibility at the contact and compression-extension phases does not improve the model at all compared with a rigid body assumption. It surely complicates the analysis (see [519] [522]), but it is not sure at all that
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assumption can be considered as legitimate due to the very short time of the multiple
 collisions. The authors also show that a compliant model of the system, composed
 of a spring+damper obstacle and a rod modeled by several elements related with
 springs and dampers, provides numerical results close to the experimental ones. Do
 these experimental results tell us that rigid body impact laws should be abandoned?
 Certainly this is not the case, as recognized by the authors themselves [223]. In fact
 the lesson is that when noncentral impacts are to occur, then great caution must
 be taken in applying Newton’s restitution law. In the case of the falling slender rod
 tested in [516], it may be argued that if the dropping surface was softer, then the
 restitution coefficient would not have varied as much as reported, mainly because
 the vibrations would have been reduced. Note however that we arrive here at a
 paradoxical situation: rigid body assumptions work better if the bodies are less
 rigid!

Remark 4.10 It is note worthy that the works in [234] [274] [519] and [36] (and
 this is a general remark that applies to all ”non-totally” rigid studies) consider
 only the interaction forces in the process, by assuming either that the colliding
 bodies evolve freely in space [234] [519], or applying implicitly a distributional-
 like derivation of the dynamics [36] [57] [274] [502] [572]. As we saw in chapter 2,
 the external forces can have a non-negligible influence on the well-posedness of the
 some impact problems when one wants to study what happens in the limit when
 the bodies become rigid. Moreover from a more mechanical point of view, it is clear
 that adding an external excitation must modify drastically the impact process in
 most of the cases. We therefore tend to believe that there is a little inconsistency to
 suppose that all external forces are negligible (which is formally equivalent to the
 assumption that interaction forces are Dirac measures) and at the same time to treat
 the bodies as compliant. This however does not preclude possible good accuracy
 of such methods in certain practical cases. Another point of view is to argue that
 since in the limit case we know that only the interaction forces play a role at impact,
 it is sufficient to consider compliant problems $P_n$ with no external action, as long
 as the goal is to prove that $\{P_n\}$ converges to some nonsmooth problem $P$. It is
 important however to note that the bodies displacements during the collision can
 have a non-negligible influence on the process outcomes. We refer the reader to
 section 5.2 where the 3-balls example illustrates this fact.

Mixed analysis

We briefly describe now studies that start with a rigid-like formulation of dynamics,
 i.e. assume ”infinitely large” impulsive forces [49] [572], zero duration percussion
 [505] [572], and assume at the same time some compliance of the surface of con-
 straints, so that impact duration is not zero, compression and expansion phases

\[ \text{it is something useful. If, anyway, one has the choice between two macroscopic models that both yield approximate predictions, the simplest one is the best one! In conclusion, it is not clear at all whether Poisson’s assumption is conceptually more satisfying than Newton’s one.} \]
exist and Poisson's rule can be applied. The difference between these works and the differential ones mainly lies in that the authors introduce their study as if it were rigid-bodies dynamics, although they could avoid such assumptions by simply assuming no external forces and a constant position during impact.

In [572] Wang and Mason analyze planar impacts with friction, and start from rigid bodies hypothesis, in particular zero measure impact intervals. Then it is assumed that compression and expansion phases exist during the process, and Poisson's rule is used: although this procedure might represent the limit of a compliant problem, this is not shown in [572]. Routh's graphical method is used to determine the total impulse \(^{(24)}\). The work in [572] is discussed by Stronge (see the same reference in our bibliography), who argues that in case of eccentric collisions or velocity slip reversal, the coefficients as introduced by [572] depend on initial orientation of the bodies, friction, \(v_{r,t}(0)\) and internal sources of dissipation. Thus Newton's and Poisson's rules cannot be constant and are therefore useless in practice since they depend on too many conditions. According to Stronge only \(e_\ast\) is useful. These facts are however noted by the authors [572] not to contradict their results. [506] makes similar basic hypothesis about rigid dynamics. The impact process is analyzed at 0 and \(t_f\) only. \(T_L\) is expressed as a function of restitution coefficient, final and initial velocities, and normal impulse \(P_{n,c}\) during compression phase (similarly as in [572] the impulse is split into the two phases); two expressions are given depending on whether \(v_{r,t}(0) \neq 0\) (initial slip) or no slip occurs on \([0, t_f]\). Several examples are given to illustrate the results. In [3] the authors show that the coefficient of restitution for eccentric impacts depend on an "effective" approach velocity (that is the ratio of \(v_{2,n}(0)\) and a coefficient depending on kinematics and friction, similar to the "effective" masses \(m_j\) and \(n_j\) in [519]), thus extending the works in [99] [250] [528] on dependence of \(e\) on the approach velocity. [233] introduces in the constraints function \(f(q)\) a stochastic term \(g(q) \ll 1\) that represents the microstructure of the surface. A Kelvin-Voigt model is used for the interaction forces. Then a probabilistic restitution rule for \(q(t_f^+)\) as a function of \(g\) and preimpact velocities is deduced, which reduces to Newton's rule for smooth surfaces. The equivalent coefficient of friction \(\mu\) is also proved to be dependent of \(M(q)\) and preimpact velocities. No mention is made in [233] about energetical behaviour of the proposed model.

---

\(^{(24)}\)Routh's method consists of studying 2-dimensional impact processes by plotting the impulsion \(P\) as a function of \(p_n\) and \(p_l\). It is assumed that a phase of compression and a phase of restitution exist. It can be shown that there exists 3 straight lines in the \(p_l - p_n\)-space that represent maximum compression, termination of the shock process, and the values of \(p_l\) and \(p_n\) corresponding to sticking. Then the shock process can be graphically deduced, depending on the initial velocity. A clear summary of Routh's method can be found in [572]. Unfortunately, Routh's method cannot be extended to the 3-dimensional case. This is the reason of subsequent works like Keller's [274], as we mentioned above.
4.2. PERCUSSION LAWS

4.2.6 Dynamical equations during the collision process: Keller's shock equations

By doing the mixed assumptions of rigidity but with nonzero impact duration, it is possible to derive the\(^{(25)}\) differential system that represents the evolution of the tangential velocity, via the time scaling using the normal impulse that is currently used in impact analysis [523] [274] [234], see subsection 4.2.4. Since those dynamical equations are based on a differential-like approach, one may start by rewriting (4.34) and (4.33) as:

\[
\begin{aligned}
\frac{d\Omega}{dt}(t) &= \mathcal{I}_i^{-1} R_i^T(t) F_i(t) \\
\frac{dX_i}{dt}(t) &= \frac{1}{m_i} F_i(t)
\end{aligned}
\]  

(4.51)

Then one makes the assumption that the duration of the shock process is small and that the shock occurs on \([t_k, t_k + t_f]\). Let us make the time scaling \(t' = \frac{t - t_k}{t_f}\). Then we obtain since \(dt' = \frac{dt}{t_f}\):

\[
\begin{aligned}
\frac{d\Omega}{dt'}(t'f') &= t_f \mathcal{I}_i^{-1} R_i^T(t'f') F_i(t'f') \\
\frac{dX_i}{dt'}(t'f') &= \frac{1}{m_i} F_i(t'f')
\end{aligned}
\]  

(4.52)

where \(t'' = \frac{t_k + t'}{t_f}\). When \(t_f \to 0\), \(R_i(t') \triangleq R_i(t'f') = R_i(t_k + t_f t') \to R_i(t_k) = R_i(t' = 0) = R_i(0)\). Note that \(t'(t_k) = 0\), \(t'(t_k + t_f) = 1\). Hence by integrating (4.52) between 0 and 1 one gets:

\[
\sigma_{\Omega_i}(t_k) = \Omega_i(t_k + t_f) - \Omega_i(t_k) = \mathcal{I}_i^{-1} R_i^T(t_k) P_i(t_k + t_f)
\]  

(4.53)

that we can rewrite in \(t'\)-scale as:

\[
\sigma_{\Omega_i}(0) = \mathcal{I}_i^{-1} R_i^T(0) P_i(1)
\]  

(4.54)

with \(P_i(1) = P_i(t = t_k + t_f) = \int_0^1 t_f F_i(t') dt' = P_i(t' = 1) \) \(^{(26)}\). Similarly:

\[
\sigma_{X_i}(t_k) = \frac{1}{m_i} P_i(t_k)
\]  

(4.55)

or equivalently

\[
\sigma_{X_i}(0) = \frac{1}{m_i} P_i(1)
\]  

(4.56)

Let us now focus on the relative velocity of the bodies at the contact point \(A\), i.e.

\[
v_r = V_{A_1} - V_{A_2}
\]  

(4.57)

---

\(^{(25)}\)Here we write the and not a. The reason is made clear next.

\(^{(26)}\)Implicitely one supposes that \(t_f F_i \neq 0\) as \(t_f \to 0\). This allows to consider all other external actions \(F_e\) as negligible during the shock process, since for them \(t_f F_e \to 0\) as \(t_f \to 0\).
Supposing a compression-extension shock process, one finds that \( v_{1,n}(t_c) - v_{2,n}(t_c) = v_{r,n}(t_c) = 0 \), where \( t_c \) is when the compression phase ends. Poisson's restitution rule gives \( p_{i,n}(1) = (1 + e)p_{i,n}(t_c) \), with \( t_c = \frac{t_i - t_f}{t_f} \). From the fact that \( p_{i,n}(t') = \int_0^t F_{i,n}(u)du \), one has \( \frac{dp_{i,n}}{dt}(t') = F_{i,n}(t') \). Assuming that \( v_{r,t_2} = 0 \) (\( t_{11} \) is chosen colinear and in the same direction as the relative tangential velocity \( v_{r,t_1} \)) and applying Coulomb's friction law \( |F_{i,t}| = \mu|F_{i,n}| \), i.e. \( \frac{F_{i,t}}{|F_{i,n}|} = -\mu t_{11} \) when there is sliding, and \( F_i \) lies inside the friction cone when sticking occurs), one finds:

\[
P_i(1) = p_{i,n}n_1 - \mu \int_0^{p_{i,n}(1)} t_{11}(p_{i,n})dp_{i,n} \quad (4.58)
\]

where one notes that since we are dealing with a differential analysis, \( t_{11} \) is allowed to vary on \([t_k, t_k + t_f]\) (or on \([0, 1]\) in \( t' \)-scale), whereas \( n_1 \) is assumed to remain constant (\(^{27}\)). Introducing Poisson's rule into (4.58) it follows that:

\[
P_i(1) = (1 + e)p_{i,n}(0)n_1 - \mu \int_0^{(1+e)p_{i,n}(t_c)} t_{11}(p_{i,n})dp_{i,n} \quad (4.59)
\]

One can now introduce (4.59) into (4.54) and (4.56) to obtain the expression relating \( \sigma_{\eta,0}(0) \) and \( \sigma_{\eta,0}(0) \) to Poisson's coefficient \( e \), \( \mu \) and \( p_{i,n}(0) \). Using (4.32) one also finds that:

\[
\sigma_{V_{\eta,0}}(0) = \left[ \frac{1}{m_i} I_3 + R_i(0)I_i^{-1}R_i(0)^T \right] P_i(1) \quad (4.60)
\]

and one can use (4.57) to relate \( \sigma_{\eta,0}(n) \) to \( e \), \( \mu \) and \( p_{i,n}(0) \). Now from (4.57) and the time-equivalent of (4.32), i.e.

\[
\frac{dV_{\eta_i}}{dt}(t) = \left[ \frac{1}{m_i} I_3 + R_i(t)I_i^{-1}R_i(t)^T \right] F_i(t) \quad (4.61)
\]

one finds by introducing \( dt' = \frac{dt}{t_f} \), \( dp_{i,n} = F_{i,n}dt' \) and Coulomb's friction rule into (4.61):

\[
\frac{dV_{\eta_i}}{dp_{i,n}}(t') = \left[ \frac{1}{m_i} I_3 + R_i(t')I_i^{-1}R_i(t')^T \right] (F_{i,t_11} + n_1) \quad (4.62)
\]

Let us denote \( \frac{1}{m_i} I_3 + R_i(t')I_i^{-1}R_i(t')^T = M_i^{-1}(t') \). Then it follows that \( \frac{dp_{1,n}}{dp_{2,n}} = M_1^{-1}(F_{1,t_11} + n_1) - M_2^{-1}(F_{2,t_11} + n_1) \). Recalling that \( dp_{1,n} = -dp_{2,n} \), \( F_{1,t_1} = -F(2, t_1) \) from Newton's principle of mutual actions, one obtains Keller's shock equations:

\[
\left\{ \begin{array}{l}
dp_{1,n} = \left[ \sum_{i=1}^{2} M_i^{-1}(F_i + n_1) \right]^T n_1 \\
dp_{2,n} = \left[ \sum_{i=1}^{2} M_i^{-1}(F_i + n_1) \right]^T t_{11}
\end{array} \right. \quad (4.63)
\]

whith \( p_n \triangleq p_{1,n}, F_i \triangleq F_{i,t_11}, v_{r,t} \triangleq v_{r,t_1}, M_i \triangleq M_i(t' = 0) \). As we pointed out earlier, Stronge [523] derives also these equations (see [523] equations (12)-(14)), and

\(^{27}\)Is such an assumption realistic?
4.2. PERCUSSION LAWS

studies from them the dynamics of a sphere or of a spherical pendulum colliding on a rough plane. In fact from (4.63) and by developing further the equations one can obtain the dynamics under the form:

\[
\begin{bmatrix}
\frac{dv_{t_1}}{dp_n} \\
\frac{dv_{t_2}}{dp_n}
\end{bmatrix} = M^{-1} \begin{bmatrix}
-\mu \cos(\zeta) \\
-\mu \sin(\zeta) \\
1
\end{bmatrix}
\] (4.64)

where \( \zeta \) is the angle made by \( v_{r,t_1} \) and \( v_{r,t_2} \) in the tangent plane \( (t_1, t_2) \), i.e. \( \zeta = \arctan(\frac{v_{r,t_1}}{v_{r,t_2}}) \). Contrarily to Keller [274] or Routh, Stronge [523] uses the energetical coefficient \( e_* \) to compute the terminal impulse at the final separation between both bodies. From (4.48) one has:

\[
e_*^2 = -\frac{\int_{p_n(t_2)}^{p_n(t_1)} v_{r,n}(p_n) dp_n}{\int_{p_n(t_2)}^{p_n(t_1)} v_{r,n}(p_n) dp_n}
\] (4.65)

Recently further results concerning this formulation have been obtained, see [328], where it is pointed out following Keller [274] that the equations in (4.63) have the general form

\[
\begin{cases}
\frac{du}{dp_n} = g(\mu, \theta) \\
v_t \frac{d\theta}{dp_n} = h(\mu, \theta)
\end{cases}
\] (4.66)

where the tangential velocity \( v_t = v_t(\cos(\theta)t_1 + \sin(\theta)t_2) \), \( t_1 \) and \( t_2 \) form an orthonormal basis of the common tangent plane at the contact point. The conditions of slipping during the impact are discussed in [328]. In particular it happens that the three-dimensional case analyzed for instance in [523] is more complex than the planar case (treated in [519], see subsection 4.2.4). Indeed a problem that arises is that if slipping stops during the collision (which may happen if \( v_k(\theta_*^*) \) is small enough), then in a second phase of slip, the tangential velocity may not be directly opposite to the tangential percussion (whereas it is in the planar case, and for a sphere colliding a rigid wall, see example 4.1). This feature may create difficulties. Define \( \mu^* \) and \( \theta^* \) such that the corresponding tangential force \( F_t = -\mu^*(\cos(\theta^*)t_1 + \sin(\theta^*)t_2) \) is the required force to maintain sticking. Then it is only when \( \mu \rightarrow \mu^* \) that the second phase sliding velocity \( v_t \) has the same direction as \( p_t \) [328].

Notice that the nonlinear ODE in (4.63) is by assumption verified for any compliant environment, without specifying any particular model. The basic assumption is of course that the positions remain constant during the impact. Such assumptions and the use of Poisson's conjecture present the advantage of allowing to get rid of force calculations during the impact (problems related with stiff ODE's for numerical integration) [363] [523], but implies knowledge of the Poisson restitution coefficient (to be able to know the end of the collision). Moreover it possesses all the drawbacks associated to this way of modeling (There may not be a simple
CHAPTER 4. TWO BODIES COLLIDING

compression-expansion phase, displacements during the impacts can have a great importance\textsuperscript{28}). Then the authors use dynamical systems theory to investigate the different possible patterns of the velocity space, and are able to determine whether slip stops or reverses. These results need to be compared with the conditions found to identify the sliding phase in [519] [572] (2-dimensional) or [274]. An interesting and original work has been proposed [49] to study the possible outcomes of the 3-dimensional impact process using Poisson's rule, using dynamical systems and bifurcation theory.

Remark 4.11 Let us note that there may well exist two scales of observation of a body: in one scale it may be rigid, on the other one compliant. But in which proportion can they be merged to yield an accurate prediction of real percussions, and can one quantify the error due to the approximations? This is not answered to in the cited literature. Moreover, is it possible and worth merging these two scales in a model for control purposes? Force tracking control design has to rely on a particular model of the contact process (the most widely used being combinations of springs and dampers [559]), but it remains to be shown how one can get rid of such a model to design the transition phase (using macroscopic restitution rules), and at the same time prove stability of the closed-loop trajectories during the whole robotic task. As noted in [247], the way a scientist may describe contact laws depends on his research area, and on the results he desires. Control represents such an area for mechanical systems with unilateral constraints.

One path to investigate might be to assume perfect rigidity, and to consider following the mechanical studies described in sections 4.2.4 and 4.2.5 that the dynamical behaviour "during" the collisions is given by (4.63) or (4.66). Then the control of the preimpact velocities plays a crucial role in the system's evolution during a phase of collisions. In a sense, the control problem would be formulated in two scales: firstly one looks at the system as if it was perfectly rigid and treats the open-loop dynamics with appropriate models. Then the transition phase controller is designed by taking the dynamical equations in (4.63) into account (without, of course, assuming that the controller plays a role during the shock).

Remark 4.12 We have chosen the above classification because these studies pertain to the same scientific domain, i.e. mechanical engineering, and have been published in journals devoted to this field. Also it seems quite logical in the body of this book to separate rigid body from compliant body formulations, since one of our main points is to consider the former as the limit of the latter, without opposing them. Note also that both problems are equivalent in the sense that they require the use of restitution coefficients to calculate postimpact motion. Another possibility could have been to describe model-based and non-model-based studies. But as we mentioned model-based studies have been extensively described elsewhere [243] [614] and are outside the scope of our work.

\textsuperscript{28}especially when multiple impacts are considered.
Remark 4.13 Most of the preceding works deal with point contact \((29)\). Line or surface contact involve much more difficulties. The main problem is that although the interaction forces are still of distributional type (the impact times remaining of zero measure) it is not clear whether point contact rules extend to these cases. For the example of a rigid block that rotates around a corner edge and strikes the ground with all its base \([498]\), application of Newton's or Poisson's rules to the whole base with \(e = 0\) yields a solution in contradiction with experiments, in which the body continues to rotate. Similar bad conclusions hold even if the base is assumed concave (two contact points) \([385]\). The problem is easily understandable \([187]\): if two points are in contact before impact (e.g. the corner around which the block rotates), then the relative velocity at these points before impact is zero. Applying Newton's rule necessarily yields a zero relative velocity after impact, which won't be true in general. \([54]\) §6.6 discusses the difficulty associated with supposing that a force and a moment impulse act together at the base. It is worth pointing out that this problem has been given recently a solution in \([385]\), who proposes a new general contact law described in chapter 5, equation (5.71). This problem is naturally closely related to motion of complex mechanical systems subject to unilateral constraints that we discuss below. We shall come back in detail on the rocking-rolling block problem and multiconstraints in chapters 5 and 6.

Remark 4.14 The least requirement about the set of coefficients that represent an impact process is that they be constant, i.e. they do not depend on the unknowns of the problem which are the final velocities. It is clear that Newton's rule coefficient is not only material dependent but depends on initial relative velocity \([224]\) [527] [520] and on the bodies shapes, and \(\mu\) can be obtained experimentally for each set of initial conditions \([57]\) [233]. It is argued in \([519]\) that the energetical coefficient \(e_s\) is only material dependent, and that this does make it of greater interest than the other ones. Note however that it is still not clear whether the conclusions about dependence of the restitution coefficients on parameters stem from the nature of the considered problem or if they should apply to all cases (elastic impact with any model, rigid impact). For instance, \([244]\) argues that the choice of a constant tangential restitution as proposed in \([54]\) yields a contradiction, because it should be process dependent. Firstly we believe that a rigorous analysis should study the compliant problem in \([244]\) in the rigid limit case to see what this coefficient tends towards. Secondly it is never claimed in \([54]\) that the coefficients should be constants, see point \(e\) in Brach's method above. Also the real problem might be to determine whether the use of this coefficient in a rigid setting could yield more prediction accuracy and keep at the same time the algebraic formulation of the shock process.

Notice en passant that by doing different hypotheses on the contact-impact problem, the behaviour during the impact phase for a given set of initial conditions should logically be different as well, since these hypotheses more or less a priori impose a model of the bodies. With this in mind, one may ask if it is realistic to search for coefficients that are independent of the physical characteristics of the colliding

\[29\] Note that moment coefficients as introduced in \([54]\) permit modelling of the effects of area.
bodies. If such a coefficient exists, then it means that it "contains" all the possible information on the impact process that one can imagine. For instance, is it possible that a coefficient be friction-independent as assumed in [519]? A more provocative question is also: is it worth seriously studying a contact-impact problem without specifying any model of the bodies? If the results agree with most experiments, then it could mean that the experimental bodies could have been accurately modelled with some physical model, and that the coefficients calculated with this model fit with the ones \textit{a priori} assumed. But it seems a quite difficult task to find a "universal" constant coefficient. To finish the discussion, if the coefficients were constant, they could not do an adequate job of modelling impacts. The reason is that because of large forces and small contact areas, impact always involve nonlinear material behaviour. Solutions of nonlinear problems typically must involve the initial conditions with material parameters to provide realistic results (for example, whether an initial velocity is high enough to cause yielding). This implies that impact coefficients must also involve initial conditions to realistically model nonlinear impact phenomenon. It is finally worth recalling that perfect rigidity does represent a particular physical model of the bodies, similarly to any compliant one. The goal of the works we have presented is to predict as accurately as possible the outcomes of an impact process. There does not seem to be a universal method, but it is more likely that the method’s (or the model’s) accuracy crucially depends on the type of problem [386]: as pointed out in [376] it is for instance unrealistic to assume rigidity when one of the bodies is very sharp at contact, because the normal direction will be very difficult to determine during the shock, because of local deformations due to high pressure. On the contrary it can be very accurate in other situations where deformations are either small or do not influence the problem parameters.

Remark 4.15 It seems that only the upperbound 0 on $T_L$ has been used as a criterion to investigate (or invent) new impact models and analysis tools. However in a particular impact process there must also exist a lowerbound for the energy loss [59], i.e. the impact itself should not dissipate so much energy that the postimpact motion cannot occur! For instance in the planar case a possible final condition is that the bodies stick at the contact point. This happens with zero normal restitution and a limiting tangential condition of zero relative velocity. In this case stating that $T_L \leq 0$ is not enough, since the minimum final energy has to be considered. Whatever coefficients are used, this provides the minimum (in the energetical sense) postimpact motion. In the three-dimensional case, the analysis has to include moments impulses that may produce a common rotational velocity as well. This problem is not mentioned in [519] [520] where the basic idea is to incorporate the natural passivity property of the shock process in the definition of the restitution rule. Related to this remark let us recall that in some simple impacting systems, the maximum loss of kinetic energy can be strictly positive. For instance, two particles of masses $m_1 = m_2 = 1$ that strike verify the dynamical equations $(\dot{x}_1^+ - \dot{x}_1^-) = -e(\dot{x}_1^- - \dot{x}_2^-)$ and $\dot{x}_1^- - \dot{x}_2^- = p_{12}$, $\dot{x}_1^- - \dot{x}_2^- = -p_{12}$. Assume that $\dot{x}_2^- = 0, \dot{x}_1^- = 1$. Then one gets $\dot{x}_1^+ - \dot{x}_1^- = -e, \dot{x}_1^+ = p_{12}, \dot{x}_2^+ = -p_{12}$. If $e = 0$, $T_L$ is maximum (see (4.41)) and we get $\dot{x}_1^+ = \dot{x}_2^+ = \frac{1}{2}$. Hence $T_{L,max} = -\frac{1}{4}$. It is not
possible to get more loss of energy at the impact using Newton's restitution rule.

**Remark 4.16** Although friction may theoretically be compensated for by bounded control during constrained phases for a manipulator with holonomic constraints, it cannot be eliminated at the impact, hence the usefulness of considering friction in the impact process.

**Remark 4.17** The rigid bodies and differential formulations do not have to be opposed: they merely possess different domains of applications. In summary there exists three main different approaches of the impact-contact problem: the classical rigid body approach with restitution coefficients, the compliant models approach (spring-dashpot or more sophisticated models), and differential-like or Keller's like approaches which are in some sense between the other two. It is clear that each one of these approaches possesses drawbacks and advantages, and it is difficult to a priori reject one of them. For instance Keller's assumptions on which the shock equations in (4.63) are based, are certainly satisfied in some instances. Then one of their usefulness is to point out the weakness of classical rigid body theory, and to highlight the differences between the three restitution coefficients $e$, $e_p$ and $e_\sigma$.

### 4.2.7 Impacts in flexible structures

Finally let us note that impacts may also occur between flexible and rigid bodies, and that a "rigid body like" impact model can be used even in this setting. Bakr, Shabana and Khulief [28] [271] study the behaviour of general multi-body systems with both rigid and flexible parts under an impulsive action, using a model based on a finite element procedure. They use a kinematic restitution coefficient to describe the shock process. As an example of possible application, they analyze the dynamics of an aircraft at touch down impact. Other works on the topic and using a restitution coefficient can be found in [214] [269] [270] [595] [596] [598]. The theoretical results are validated in [457] and [412]. The applicability of the prediction based on the use of a restitution coefficient is examined in [595] [596]. These studies show that for the considered systems, experimental and theoretical results fit quite well. Other studies use compliant models of the contact-impact process [104] [597] [272] [598]. In [598], Yigit compares numerically and experimentally three different models (restitution coefficient, spring-dashpot and Hertzian like) and concludes that they provide quite similar results. But the Hertzian-like model possesses the advantage that its coefficients are computable from the physical characteristics of the materials. Note that systems of partial differential equations with unilateral constraints have already been investigated in the mathematical literature, see e.g. [7], and feedback control of such systems has also received some attention [387] [9].

**Remark 4.18** It is convenient to visualize the definition of the three restitution coefficients (kinematic, kinetic and energetic) on a diagram as in figure 4.3. Newton's conjecture is that the ratio $\frac{\Delta \theta}{\Delta \varphi}$ is constant, Poisson's conjecture is that the ratio $\frac{\Delta \varphi}{\Delta \theta}$ is
constant, and Stronge's claim is that the ratio $\frac{\text{area}(A)}{\text{area}(B)}$ is constant. Note that such a representation relies on the basic assumption that the positions remain constant during the shock. Then one has $v_n = v_n(0) - \frac{1}{m_j} p_n$ [519]. This is precisely this type of diagram that is used in [519] to calculate the three different coefficients depending on the tangential velocity reversal. The simplest case is depicted on figure 4.3, when there is no tangential velocity reversal. Otherwise $v_n(p_n)$ is no longer a straight line, but is made of two segments whose slopes depend on the effective mass $m_j$ (see subsection 4.2.4) and the initial tangential velocity.

Figure 4.3: Normal velocity versus normal impulse.
Chapter 5

Multiconstraint nonsmooth dynamics

Until now we have mainly focused on systems submitted to unilateral constraints with codimension one boundary. In this chapter we investigate the difficulties encountered when several constraints are present. Some simple examples are used to introduce the problem. Then we present a general method, the so-called *sweeping process*, that consists of a particular way of writing the dynamical equations of systems with unilateral constraints, through specific measure differential inclusions. The chapter ends with the presentation of *complementarity* formulations of the dynamics of systems with unilateral constraints, and on the existence of solutions to such complementarity problems (related to *tangential impacts*). A brief summary of various existing methods for the numerical integration of impacting systems is given.

5.1 Introduction. Delassus’ problem

There is a problem that is more difficult than those we have investigated from now on. It consists of studying the motion of a mechanical system submitted to several unilateral constraints $f(q) \leq 0$, $f \in \mathbb{R}^m$, $q \in \mathbb{R}^n$, with $f_i(q) = 0$ for some $i \in \{1, \ldots, m\}$. These constraint inequalities define a region in the configuration space (or configuration manifold) that we shall denote as $\Phi$. As stated in [379], two crucial questions have to be answered to

- **Question 1** Starting from an instant $t_0$, with $q(t_0) \in \Phi$ and an admissible right velocity $\dot{q}(t_0^+)$ (that is roughly a velocity pointing inwards $\Phi$), determine which of the contacts $f_i(q) = 0$ persist during a subsequent interval.

- **Question 2** If some time interval during which $f_i(q) > 0$ ends at an instant $t_1$ such that $f_i(q(t_1)) = 0$, a percussion is expected to occur: determine the right velocity $\dot{q}(t_1^+)$. 
CHAPTER 5. MULTICONSTRAINT NONSMOOTH DYNAMICS

Such problems were raised initially by Delassus in [124]. We have already seen in chapter 4 how one may answer to question 2 in some particular cases of two bodies colliding: this is done by defining some restitution rule, which relates post and preimpact velocities through some coefficient of restitution. This in turn can be used in the shock dynamical equations to compute the percussion vector. The main problem concerning question 1 is the following: when there is a unique constraint, the sign of the Lagrange multiplier is sufficient to decide whether the contact persists or if the system detaches from the constraint [429] (Recall that the multiplier is a function of the system's dynamics). Now in the multiconstrained case, the problem is as follows: for some \( i \in \{1, \ldots, m\} \) and at a given instant \( t_0 \), \( f_i(q(t_0)) = 0 \) and \( \partial f_i \partial q(q(t_0)) = 0 \), and one wants to know which of them will persist, and which will not. One tries to calculate the motion under the assumption that no contact breaks (i.e. the contacts are assumed bilateral); at the same time one calculates the contact forces (i.e. the corresponding Lagrange multipliers) from the dynamical equations, and if it happens that one or several of them have a wrong direction at a given instant (with our choice \( f(q) \leq 0 \) they must be negative to keep the contact), then one concludes that some contacts have to break at this instant and the procedure has to be restarted; but contacts that break are not necessarily those that correspond to the forces in the wrong direction [124] [429] Chapter 10 §3, when \( m > 1 \). Then one has to reiterate the procedure until an acceptable solution is found. Delassus [124] has shown that in general \(^1\) only one possibility among the \( 2^m \) is satisfactory when the surfaces are frictionless. This gives an idea of the complexity of this dynamical problem. Let us consider in detail the example of Delassus (see also [429] p.303).

**Delassus' example**

The planar frictionless system is depicted in figure 5.1, and consists of a disc constrained in an angle. The system is thus constrained by two surfaces

\[
f_1(q) = ax - y - R\sqrt{1 + a^2} \geq 0 \tag{5.1}
\]

\(^1\)the example analyzed below provides a counter-example.
and

\[ f_2(q) = y - bx - Rb\sqrt{1 + b^2} \geq 0 \]  \hspace{1cm} (5.2)

where \( q = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \) is the vector of generalized coordinates with \( x \) and \( y \) the coordinates of \( G \), and \( R \) is the disk radius. \( a \) and \( b \) are the slopes of the constraints, \( a < b \).

The reactions at points \( A \) and \( B \) take the form \( p_A = \lambda_1 \begin{pmatrix} a \\ -1 \end{pmatrix} \) and \( p_B = \lambda_2 \begin{pmatrix} -b \\ 1 \end{pmatrix} \).

The Jacobian between points \( A \) and \( G \) is given by

\[
\begin{pmatrix} 1 & 0 & -\frac{R}{\sqrt{1+a^2}} \\ 0 & 1 & -\frac{Rb}{\sqrt{1+a^2}} \end{pmatrix}
\]

The Jacobian between points \( B \) and \( G \) is given by

\[
\begin{pmatrix} 1 & 0 & \frac{R}{\sqrt{1+b^2}} \\ 0 & 1 & \frac{Rb}{\sqrt{1+b^2}} \end{pmatrix}
\]

Hence the dynamical equations are given by

\[
\ddot{x} = a\lambda_1 - b\lambda_2 \\
\ddot{y} = -g - \lambda_1 + \lambda_2 \\
\ddot{\theta} = 0
\]  \hspace{1cm} (5.3)

\( g \) is the gravity acceleration. Let us study the possibility of a balanced situation where the disk keeps contacts with both surfaces, i.e. \( f_1(q) = 0 \) and \( f_2(q) = 0 \) in the future instant of time. Following for instance [429], we now compute the values of the Lagrange multipliers \( \lambda_1 \) and \( \lambda_2 \) by stating that \( \dot{f}_1(q) = a\dot{x} - \dot{y} = 0 \) and \( \dot{f}_2(q) = \dot{y} - b\dot{x} = 0 \) (this is true since contacts are supposed to persist). These equations plus (5.3) allows to calculate that

\[ \lambda_1 = \frac{gb}{a-b} < 0, \quad \lambda_2 = \frac{ga}{a-b} < 0 \]  \hspace{1cm} (5.4)

Hence the calculations yield a negative reaction at both contacting points. For point \( A \) this is logical since it is evident that the disk is going to detach from the upper constraint due to gravity action. But it is also evident that the disk will not detach from the lower constraint, despite of the fact that the reaction is computed to be negative. Hence for multiple unilateral constraints, it is not possible in general to conclude an escape from the constraint just by computing the multipliers' signs. In this example the motion can be calculated by assuming detachment at \( A \) and sliding on surface \( f_2(q) = 0 \). This example is quite simple, but imagine a complex kinematic chain with multiple constraints: then intuition may not tell the designer anything!

When friction is considered, the number of outcomes is larger since new phenomena like slip reversal can occur. To illustrate the manifold of rigid impacts mechanics possible applications, let us finally remark that the problem of multibody-multiconstraints systems may be applied to kinematic chains moving among rigid
obstacles, as well as granular materials composed of spheres [322] [386]. It is however clear that these two practical problems present too big differences to be treated exactly in the same way. Here we mainly focus on the first class of applications.

5.2 Kinematic chains with unilateral constraints

The motion of complex systems like kinematic robotic chains has been recently treated in the robotics literature [169] [187] [190] [220] [222] [334] [303] [462] [495] [496], mostly in relation with bipedal locomotion that involves motion of kinematic chains with unilateral constraints. The most advanced studies that take into account possible contact-breaks are those by Han and Gilmore in [187] and by Hurmuzlu and Marghitu in [222] and we describe them below. It precisely aims at solving the kind of problems raised in [124], using the most recent results about oblique impacts with friction between two bodies described in chapter 4.

The works by Hurmuzlu and coauthors in [220] [222] [334] aim at solving the motion problem of a n-link kinematic chain submitted to two [220] or to m unilateral constraints. It is assumed that one end strikes a constraint while the other (m - 1) endpoints are initially in contact (i.e. $f_i(q) = 0$ for $i \neq 1$ and $f_1(q) < 0$), and that all events (impacts or break of contact) are simultaneous. Roughly, the method is based on an exhaustive listing of the possible different outcomes that can occur after the impact (for instance the points initially in contact may detach from the surface, slip or remain stationary, there may be velocity reversal or stop at the colliding end ...). Note that this seems the only possible way to proceed according to the pioneering works on the subject in [124] [429]. Then the authors use the developments on collisions between two bodies developed in [56] [54] [274] to study whether the predicted outcome fits within the proposed theory or not. Only the algebraic method [56] is considered in [220]. The differential and algebraic treatments are considered and compared through a numerical example in [222], as well as the three different restitution coefficients (Newton, Poisson, Stronge). The results may significantly diverge one from each other when friction is present, still demonstrating that the available "rigid" methods (see subsection 4.2.2) are not to be considered as the limit of the "compliant" ones (see subsection 4.2.4 and 4.2.5). As expected, the differential approach yields a more complicated 15-steps algorithm, whereas the algebraic one relies on a 7-steps algorithm. Of course since the methods proposed in [220] and [222] are based on possibly non well-settled theories of impact processes, they inherently contain the possible erroneous or approximative conclusions these theories yield. The work in [334] extends the method in [222] for three-dimensional collision problems.

Han and Gilmore [187] propose analytical computer oriented method to analyze multiple-body impacts, including possible friction and sliding. The usual assumptions for "mixed" analysis are done (see subsections 4.2.4 and 4.2.5), and a graphical Routh analysis of the possible behaviours during percussion is provided, based on Poisson's and Coulomb's rules. The benchmark example of a rod falling on a rigid
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ground is treated. The results show that for some values of the friction coefficient, the kinetic energy at impact increases as the restitution coefficient does, a phenomenon pointed out in [56]. The authors analyze the outcomes in multibody systems when some contacts may break, due to *internal impacts* (see definition below). A computational algorithm is presented, based on a particular topological description of the system: the distance $k$ between contact-impact points is chosen to be the minimum number of bodies that separate a given point and the prespecified reference point. The algorithm uses the impact analysis between two bodies developed in [187] to calculate, for each $k$ (starting at $k = 0$) the post-impact motion. Then an exhaustive procedure that considers all possible outcomes during the impact process is given. An experimental validation is given that shows quite good matching with simulation.

Let us note that it is not stated neither in [222] nor in [187] that the proposed algorithms yield a solution in all cases, and if it does whether it is unique or not. As the simple 3-balls system shows in example 5.1, uniqueness cannot be expected in general. If it is not unique, which criterion should be used to detect the good one? If the rigid body assumption is not relaxed, all admissible solutions have to be treated with the same consideration. In fact rigidity precludes any particular choice. Note that Delassus's result for the frictionless case is that uniqueness holds "in general". The three balls example shows that some singular cases are outside the "generality".

Are these studies equivalent, i.e. can they be used to solve the same problems? In fact, in [222] is developed an algorithm for a $n$-degree-of-freedom planar system with multiple unilateral constraints, whereas in [187] the authors deal with a planar system of $n$ bodies in contact through their vertices and/or faces, and is closed to granular materials studies. It seems that although both systems pertain to the same class of mechanical systems, their different topologies (in the mechanical sense) yields different algorithms for the postimpact motion determination.

Two classical examples are treated to explain the possible outcomes of some impacting systems with ambiguous solutions. We provide them here in detail since we shall need them later when we deal with generalized restitution rules in the configuration space, with the kinetic metric.

5.2.1 The striking balls examples

Example 5.1 (3-balls system) Let us consider the system depicted in figure 5.2. The three balls (or spheres, or particles) are sliding horizontally. There is no dissipation between the balls and the ground. Han and Gilmore [187] introduce the notion of *internal* and *sequential* impacts: internal impacts are impacts that occur between two bodies previously in contact, i.e. which occur in fact through an internal transmission inside the bodies, and such that they create detachment. The authors also assume the possibility of a certain chronology for the possible impacts occurring in the system, hence sequential impacts. Let us explain how solutions (i.e.
postimpact velocities) can be found. The shock dynamical equations are given at the shock instant $t_k$ by

$$
\dot{q}_1(t_k^+) - \dot{q}_1(t_k^-) = -p_{12}
$$

$$
\dot{q}_2(t_k^+) - \dot{q}_2(t_k^-) = p_{12} - p_{23}
$$

$$
\dot{q}_3(t_k^+) - \dot{q}_3(t_k^-) = p_{23}
$$

The masses are taken equal to one for simplicity, and the preimpact velocities are chosen as $\dot{q}_1(t_k^-) = 1$, $\dot{q}_2(t_k^-) = \dot{q}_3(t_k^-) = 0$. It is supposed no energy loss ($T_L = 0$) at impacts. Two postimpact sets of velocities are computed and are given by

$$
\dot{q}_1(t_k^+) = -\frac{1}{3} \quad \dot{q}_2(t_k^+) = \dot{q}_3(t_k^+) = \frac{2}{3}
$$

and

$$
\dot{q}_1(t_k^+) = \dot{q}_3(t_k^+) = 0 \quad \dot{q}_2(t_k^+) = 1
$$

The solution in (5.6) can be found by applying Newton's restitution rule with $e = 1$ between bodies 1 and 2 (i.e. $\dot{q}_1(t_k^+) = -1 + \dot{q}_2(t_k^+)$), and between bodies 2 and 3 (i.e. $\dot{q}_2(t_k^+) = \dot{q}_3(t_k^+)$), and assuming a nonzero $p_{23}$ (i.e. implicitly assuming a nonzero $\dot{q}_3(t_k^+)$). The solution in (5.7) can be found by assuming no shock between bodies 2 and 3, i.e. $p_{23} = 0$. Now notice that (5.6) can be set as definitive since postimpact motion is possible: the first body rebounds and the other two remain stuck. But solution 2 is not feasible between bodies 2 and 3: that problem is overcome in [187] by assuming a second impact between bodies 2 and 3. Applying Newton's rule between bodies 1 and 2 and bodies 2 and 3 yields another nonfeasible solution. But assuming there is no impact between 1 and 2 (i.e. $p_{12} = 0$) yields a feasible motion. This solution is then given by

$$
\dot{q}_1(t_k^{++}) = \dot{q}_2(t_k^{++}) = 0 \quad \dot{q}_3(t_k^{++}) = 1
$$

The superscript $^{++}$ is to distinguish the impacts chronologically. This solution is a possible motion: bodies 1 and 2 remain stuck, body 3 moves to the right.

In fact the above reasoning relies on three rules:

$^2$Notice that is is logical to associate a restitution coefficient and at the same time assume a nonzero percussion. The contrary would yield a contradiction, since a continuous velocity implies a zero impulsion, see chapter 4.
• i) The kinetic energy loss at impact is zero.

• ii) The postimpact velocity must assure a feasible motion, i.e. point inwards the domain inside the constraints.

• iii) Let us denote $q_i$ and $q_{i+1}$ the coordinates of two successive balls. Then if $\dot{q}_i(t^-) > \dot{q}_{i+1}(t^-)$, the percussion between these two bodies $p_{ij} \neq 0$. If $\dot{q}_i(t^-) < \dot{q}_{i+1}(t^-)$, $p_{ij} = 0$. If $\dot{q}_i(t^-) = \dot{q}_{i+1}(t^-)$, then two possibilities must be tested: either $p_{ij} = 0$ or $p_{ij} \neq 0$.

It can be shown that due to the particular choice of the initial conditions, i implies that the restitution coefficients between the balls is equal to 1. ii allows to decide at each step whether a computed velocity is admissible or not. iii is a fundamental rule which allows to decide of the form of the percussion vector. It can be shown that in this particular example, the algorithm has a finite number of iterations, and that the only two possible postimpact velocities are the ones in (5.6) and (5.8). When an admissible velocity has been found, it is considered as definitive. But all possible paths have to be tested. A detailed treatment of this example through the Han and Gilmore algorithm can be found in [96].

Remark 5.1 A similar analysis as the one in [187] for systems composed of several spheres colliding (known sometimes as Newton's cradle, may be because it has the appearance of swinging) has been performed in [251], where sequential impacts are also introduced. The phenomenon is treated as a succession of simple impacts in [251], so that no problems due to multiple impacts occur (see chapter 6 for more details on multiple impacts).

Remark 5.2 What we call in the following the 3-balls problem corresponds to Newton’s cradle. But what we denote as the 2-balls problems is different from Newton’s cradle, since it is composed of two balls striking a rigid wall.

Thus the Han and Gilmore algorithm yields two possible solutions for the postimpact velocities, and it is a priori impossible to decide which one is the right one, just relying on rigid body theory. It is easy to observe that in practice, such 3-balls systems evolve closely to the solution in (5.8). But this solution is not really the experimental one, because one also easily observes that although the third ball detaches quickly from the second one, the second and the first one possess nonzero postimpact velocity, and do have a motion after the shock. The balls are currently made of hard material (iron) so that the rigid body assumption can be considered to be valid in this case. Can we explain such phenomenon still supposing zero energy loss and zero impact duration? To the best of our knowledge, the only way to solve such rigid body undeterminancy is to add some compliance to the bodies that contact.
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Approximating compliant problems

Which one of (5.6) and (5.8) is the real motion however? As always in mechanics of rigid body, indeterminate or inconsistent situations can be avoided by studying some approximating compliant problems. Let us calculate the motion for both approximating compliant systems depicted in figures 5.3 and 5.4. In the first system, the percussion between bodies 1 and 2 is rigid. But the contact model between bodies 2 and 3 is a spring of stiffness $k > 0$. We apply Newton's rule with $e = 1$ between 1 and 2, which provides us with new initial conditions $q_1(t_k^+) = 0$, $q_2(t_k^+) = 1$, $q_3(t_k^+) = 0$ for the motion between 2 and 3. Then from the dynamical equations it can be calculated that the postimpact motion is

$$
q_2(t) = \frac{t}{2} - \frac{1}{2\sqrt{2k}} \sin(\sqrt{2kt}) 
$$

(5.9) and

$$
q_3(t) = \frac{t}{2} + \frac{1}{2\sqrt{2k}} \sin(\sqrt{2kt}) 
$$

(5.10)

It can thus be deduced from (5.9) and (5.10) that bodies 2 and 3 detach at $t_1 = \frac{\pi}{\sqrt{2k}}$.

The impulsion of the interaction force between bodies 2 and 3 is then given by

$$
p_{23}(t_1) = \int_0^{t_1} k(q_2(t) - q_3(t)) dt = 1 
$$

(5.11)

This value is independent of $k$. Now as $k \to +\infty$, $t_1 \to 0^+$ and it follows from (5.9) and (5.10) that $q_2(t_1) \to 0$, $q_3(t_1) \to 1$. Hence the solution of this approximating problem converges towards (5.8).

\textsuperscript{3}Note that one may argue that the approximation is in fact the rigid model, since in reality bodies deform at contact. The word approximation is to be understood here in the mathematical sense, as a sequence of problems whose solutions converge towards solutions of a limit problem.

\textsuperscript{4}We take for simplicity of notations $t_k = 0$. 

---

Figure 5.3: Approximation of the 3-balls system (second contact compliant).

Figure 5.4: Approximation of the 3-balls system (first contact compliant).
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Consider now the second approximating problem in figure 5.4. This time we replace the rigid percussion between bodies 1 and 2 by a spring-like contact model with stiffness $k$, whereas the contact between 2 and 3 is rigid. The dynamical equations of the system when contact between the bodies is established are given by

$$
\begin{align*}
\dot{q}_1 &= k(q_2 - q_1) \\
\dot{q}_2 &= k(q_1 - q_2) + p_{23} \\
\dot{q}_3 &= -p_{23}
\end{align*}
$$

(5.12)

Note that $p_{23}$ can represent either a bounded contact force (time-function), or an impulsive force if there is a shock between balls 2 and 3. Initially $p_{23}(0) = 0$. Assume there is an interval $[0, \delta]$, $\delta > 0$, such that $p_{23} \equiv 0$ on $[0, \delta]$. Then the dynamical equations yield on $[0, \delta]$

$$q_1(t) = \frac{t}{2} + \frac{1}{2\sqrt{2k}} \sin(\sqrt{2k}t)$$

(5.13)

and

$$q_2(t) = \frac{t}{2} - \frac{1}{2\sqrt{2k}} \sin(\sqrt{2k}t)$$

(5.14)

Hence $q_2(t) = \frac{1}{2} - \frac{1}{2} \cos(\sqrt{2k}t)$, and on $[0, \delta]$, $q_2(t) > 0$ for $\delta$ small enough. But now from the assumption that $p_{23} \equiv 0$ and the initial conditions, it follows that $q_3 \equiv 0$ on $[0, \delta]$. This yields a contradiction. Hence we must have $p_{23} < 0$ on a nonzero interval after the contact has been made between 1 and 2 ($p_{23}$ is the action of 3 on 2). Now the next time such that the bodies 2 and 3 may detach is $t = T$ such that $p_{23}(T) = 0$. Let us denote $z = q_1 - q_2$. From the dynamical equations, it follows that if $p_{23} = 0$, then $\dot{z} + 2kz = 0$ and $\ddot{z} + \frac{3}{2}kz = 0$. It follows that necessarily at $t = T$ one gets $z = 0$, i.e. $x_1(T) = x_2(T)$ and $p_{23}(T) = 0$. Now if there is detachment between 2 and 3 (i.e. the bodies no longer touch), then $\dot{q}_3(t)$ must take a larger value than $\dot{q}_2(t)$ at a certain time $t \geq T$. This is impossible since this would imply a positive gain of kinetic energy for body 3 from a source else than body 2. Moreover at $t = T$ both bodies are still in contact with the same velocity, and then they just touch one each other with zero interaction. Hence $p_{23}(t) = 0$ on $[T, T + \delta]$ for some $\delta > 0$.

Now on $[0, T]$ since bodies 2 and 3 interact it follows that $\ddot{y} = 0$, where $y = q_1 + 2q_2$ and $y(0) = 0$, $\dot{y}(0) = 1$. At the same time $\ddot{z} + \frac{3}{2}kz = 0$, and $z(0) = 0$, $\dot{z}(0) = 1$. It can be calculated that

$$
\dot{q}_1(t) = \frac{1}{3} + \frac{2}{3} \cos\left(\frac{3}{2}kT\right)
$$

(5.15)

and

$$
\dot{q}_2(t) = \frac{1}{3} - \frac{1}{3} \cos\left(\frac{3}{2}kT\right)
$$

(5.16)

From (5.15) and (5.16) bodies 1 and 2 detach at $t_1 = \sqrt{\frac{2}{3k}} \pi$. Note that on $[T, t_1]$ the same reasoning as above applies to prove that $p_{23} \leq 0$ on this interval. Hence the
motion is given by (5.15) and (5.16) on \([0, t_1]\). Now it is easily calculated from (5.15) and (5.16) that \(\dot{q}_1(t_1) \rightarrow -\frac{1}{3}\) and \(\dot{q}_2(t_1) = \dot{q}_3(t_1) \rightarrow \frac{2}{3}\) as \(k \rightarrow +\infty\), whereas \(t_1 \rightarrow 0^+\). Therefore the solution of this approximating problem converges towards (5.6). Note that the interaction between bodies 2 and 3 is nonzero as long as the interaction between 1 and 2 is nonzero. When 1 and 2 have detached, then \(p_{23}\) becomes equal to zero.

Consider finally the general approximating sequence of compliant problems as depicted in figure 5.5. The dynamical equations of the system are given by

\[
\begin{align*}
\ddot{q}_1 &= \begin{cases} 
0 & \text{if } q_2 \geq q_1 \\
-k_1(q_1 - q_2) & \text{if } q_2 < q_1
\end{cases} \\
\ddot{q}_2 &= \begin{cases} 
0 & \text{if } q_2 \geq q_1 \text{ and } q_3 \geq q_2 \\
-k_1(q_2 - q_1) & \text{if } q_2 < q_1 \text{ and } q_3 \geq q_2 \\
-k_2(q_2 - q_3) & \text{if } q_2 \geq q_1 \text{ and } q_3 < q_2 \\
-k_1(q_2 - q_1) - k_2(q_2 - q_3) & \text{if } q_2 < q_1 \text{ and } q_3 < q_2
\end{cases} \\
\ddot{q}_3 &= \begin{cases} 
0 & \text{if } q_3 \geq q_2 \\
-k_2(q_3 - q_2) & \text{if } q_3 < q_2
\end{cases}
\end{align*}
\]

(5.17)

We can represent in this example the difference of compliance between both contact points with a positive constant \(\alpha\), i.e. \(k_2 = \alpha k_1\) and \(k_1\) is denoted as \(k\). We shall analyze in detail this more general case in chapter 6, subsection 6.5.4, where we shall also present the two-balls case (the third ball 3 is replaced by a rigid wall).

It is therefore apparent from the two studied approximating problems that the postimpact motion crucially depends on where the compliance is placed. In subsection 6.5.4 we shall consider two compliant contacts and study convergence when both stiffnesses grow unbounded at different rates (due to the constant \(\alpha\) in (5.17)). This study will show that neglecting the bodies' displacements during the collision process may yield erroneous conclusions. Indeed one could think of analyzing such systems as proposed for instance in [274] or [363], see subsections 4.2.4 and 4.2.5: one associates a macroscopic restitution Poisson coefficient to each contact. Via the
normal-impulse time-scaling (see subsection 4.2.4), it is then possible to integrate
the velocities during the collision. Our example clearly corresponds to a Poisson
coefficient $e = 1$, since there is no energy loss during the shocks. Then adopting
such method, how can we determine the different outcomes corresponding to dif-
ferent stiffnesses? This is actually a rigid body formulation, not an approximating
problem. Obviously the consideration of non-constant position during the shock
implies the consideration of a particular model of contact. If in practice one knows
that the bodies, which are nevertheless modeled as rigid, do actually possess more
compliance at one place than at another, then one may choose preferably one solu-
tion rather than the other one. It would be interesting to derive a general method,
consisting of using rigid body dynamics (for simplicity of the dynamical equations),
and introducing some weighting factors for possible flexibilities (or more exactly
that would represent the relative quantity of flexibility between the bodies), that
would yield a particular solution for the postimpact values. One should find out
some general rules allowing to state that if (small) flexibilities are placed at one
place or at another, then motion will probably occur in a certain manner.

We now turn our attention to a second classical problem, where the third ball is
replaced by a fixed obstacle.

Example 5.2 (2-balls system) Let us consider the system depicted in figure 5.6.
Applying the algorithm described above [187] with the same initial data ($\dot{q}_1(t^+_{k-}) = 1,$
$\dot{q}_2(t^+_{k-}) = 0, q_2(t_k) = 0$), it is possible to calculate two sets of postimpact velocities

$$\dot{q}_1(t^+_{k}) = -1 \quad \dot{q}_2(t^+_{k}) = 0$$

which is feasible and

$$\dot{q}_1(t^+_{k-}) = 0 \quad \dot{q}_2(t^+_{k-}) = 1$$

which is not feasible. From (5.19) another set is computed as

$$\dot{q}_1(t^{++}_{k-}) = 0 \quad \dot{q}_2(t^{++}_{k-}) = -1$$

which is still not feasible. Hence another iteration is necessary which yields

$$\dot{q}_1(t^{+++}_{k}) = -1 \quad \dot{q}_2(t^{+++}_{k}) = 0$$

Now (5.21) yields a feasible postimpact motion. In this case there is only one solution
to the algorithm proposed in [187].
As for the three-balls case, one can study a simple sequence of approximating problems, as depicted in figure 5.7. The dynamical equations are given by

\[
\begin{align*}
\dot{q}_1 &= \begin{cases} 
0 & \text{if } q_1 < q_2 \\
 k(q_1 - q_1) & \text{if } q_1 \geq q_2
\end{cases} \\
\dot{q}_2 &= \begin{cases} 
0 & \text{if } q_1 < q_2 \text{ and } q_2 < 0 \\
 -\alpha kq_2 & \text{if } q_1 < q_2 \text{ and } q_2 \geq 0 \\
k(q_1 - q_2) & \text{if } q_1 \geq q_2 \text{ and } q_2 < 0 \\
k(q_1 - q_2) - \alpha kq_2 & \text{if } q_1 \geq q_2 \text{ and } q_2 \geq 0
\end{cases}
\end{align*}
\] (5.22)

where it is supposed that \( q_1 = 0 \) and \( q_2 = 0 \) corresponds to the position where the springs are at rest.

It is then possible to lead similar calculations on several approximating problems as for the three balls case. Let us for instance consider the case where the contact between balls 1 and 2 is rigid, whereas the contact between the wall and ball 2 is compliant, with stiffness \( k \). Then \( \dot{q}_1(t^*_k) = 0 \) and \( \dot{q}_2(t^*_k) = 1 \). The dynamics after the shock are given by \( \ddot{x}_2 = -kx_2 \) with initial conditions \( q_2(t_k) = 0, \dot{q}_2(t^*_k) = 1 \). Thus \( q_2(t) = \frac{1}{\sqrt{k}} \sin(\sqrt{k}t) \). Detachment between ball 2 and the spring occurs when \( q_2(t_1) = 0, t_1 > t_k = 0 \), and at the same time an impact occurs between the balls. One gets \( t_1 = \frac{\pi}{\sqrt{k}} \), so that \( \dot{q}_1(t^*_1) = 0 \) and \( \dot{q}_2(t^*_1) = -1 \), and \( \dot{q}_1(t^*_1) = -1, \dot{q}_2(t^*_1) = 0 \). Notice that these final velocities are independent of the value of \( k \). Now consider the approximating problem where the contact between ball 2 and the wall is rigid, whereas it is compliant with stiffness \( k \) between the two balls. Then obviously the force exerted by the ball 1 on ball 2 is such that ball 2 remains in contact with the wall. It is easy to see that \( x_1(t) = \frac{1}{\sqrt{k}} \sin(\sqrt{k}t) \), and the final velocities are the same as previously.

The more general case when there is compliance at both contacts will be analyzed in subsection 6.5.4.
5.3 Moreau's sweeping process

We now describe a general tool that aims at solving the problem of multibody systems with multi-unilateral constraints. Its main feature compared to any other work we are aware of, is that it was originally designed to encompass this problem in its generality, i.e. the mathematical as well as mechanical and numerical-treatment sides are considered. It is of course still an active topic of research. It is noteworthy that this formulation may also encompass other mechanical problems such as water falling in a cavity, plasticity and the evolution of elastoplastic systems [377].

5.3.1 General formulation

Nonsmooth dynamics of multibody mechanical systems submitted to several unilateral constraints is analyzed in [378] [379] [381] [366] through the use of the so-called "sweeping or Moreau's process" (see [379] §7 [366] definition 1.1 chapter 1 and references therein): given a convex "moving" (i.e. time-dependent) set $C(t)$, a moving point $w(\cdot)$ is a solution to the sweeping process by $C(t)$ if

- i) $w(0) \in C(0)$
- ii) $w(t) \in C(t)$
- iii) There exists a positive measure $\mu$, relative to which the Stieltjes measure $dw$ possesses a density $w'$, i.e. $dw = w'd\mu$, and $-w' \in N_{C(t)}(w)$ (5).

Points i and ii have an obvious meaning; iii roughly means that if $w$ is on the boundary of $C(t)$ then its gradient points inwards the set $C(t)$; $N_{C(t)}(w)$ denotes the outward normal cone to $C(t)$ at $w(t)$ and can also be denoted as $\partial \psi_{C(t)}(w)$, where $\psi$ is the indicator function of $C(t)$, see definitions D.1 and D.4.

5.3.2 Application to mechanical systems

Frictionless constraints

In this part we study the application of Moreau's sweeping process to an evolution problem with unilateral constraints. All the material of this part is taken from [379] [381] [366]. In fact following closely [381] we explain how one may construct an evolution problem representing as fairly as possible nonsmooth impact dynamics, in the setting of the sweeping process.

Smooth motions It is convenient to consider first the smooth motions of the system, i.e. when $f(q) < 0$. It is also necessary to have in mind some definitions

\footnote{See appendix B for definitions.}
and notations used in the following. The mechanical system is submitted to a set of frictionless unilateral constraints $f_i(q) \leq 0$, $i = 1, \ldots, m$. It is assumed that the gradients $\nabla_q f_i(q)$ are not zero in some neighborhood of the surfaces $f_i(q) = 0$. These inequalities define in $\mathbb{R}^n$ a feasible region $\Phi$ where the system is constrained to evolve. We assume that $\Phi$ is convex in the region of interest.

The tangent space to $\Phi$ at point $q$ is $E(q)$, to which right-velocities $q(t^+)$ belong (i.e. right-derivatives of $q(t)$). In fact $E(q)$ can be identified with $\mathbb{R}^n$.

**Definition 5.1 (Tangent cone)** The convex polyhedral tangent cone $V(q)$ to the region $\Phi$ at point $q$ is given by

$$V(q) = \{ v \in E(q) : \forall i \in J(q), v^T \nabla_q f_i(q) \leq 0 \}$$  \hspace{1cm} (5.23)

where

$$J(q) = \{ i \in \{1, \ldots, m\} : f_i(q) \geq 0 \}$$  \hspace{1cm} (5.24)

Note that $V(q) = \mathbb{R}^n$ when $J(q) = \emptyset$, i.e. when $f_i(q) < 0$ for all $i$'s $^6$.

Suppose that $J(q)$ is reduced to one element $i$. Then $V(q)$ is the half-space $\{ v \in \mathbb{R}^n : v^T \nabla_q f_i(q) \leq 0 \}$. In this case the hyperplane tangent to $f_i(q) = 0$ is given by

$$T(q) = \{ v \in \mathbb{R}^n : v^T \nabla_q f_i(q) = 0 \}$$  \hspace{1cm} (5.25)

Note that the cones in definition 5.1 and in definition D.2 are not identical: they are if $q(t) \in \Phi$, but not if $q \not\in \Phi$: in fact the tangent cone is commonly taken as $\emptyset$ if $q$ is outside $\Phi$, whereas $V(q)$ is not empty in this case (see (5.24) which means that one has to take into account those positions $q$ such that the constraints are violated). As far as impact dynamics are concerned, this distinction is purely formal, because in fact $q$ will be forced to never leave $\Phi$. By doing the assumption that $q \in \Phi$ for all $t \geq t_0$ $^7$, one could therefore define $J(q)$ in (5.24) writing $f_i(q) = 0$. The definition in (5.24) can be useful in some existential results where approximating problems will imply some penetration into the constraints, see [366] chapter 3. Then one needs to define the tangent cone for points outside $\Phi$.

The *polar* cone to $V(q)$ is defined as

**Definition 5.2 (Normal cone)** The closed convex polyhedral cone $N(q)$ is given by

$$N(q) = \{ r \in E'(q) : \forall v \in V(q), v^T r \leq 0 \}$$  \hspace{1cm} (5.26)

where $E'(q)$ denotes the dual space of $E(q)$, to which generalized reaction forces belong. $N(q)$ is the outward normal cone to $\Phi$ at $q$, see definition D.3, and is

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$^6$In other words, the system is inside the domain $\Phi$ and does not touch any constraint hypersurface.

$^7$We prefer to denote the initial time as $\tau_0$ instead of $t_0$, to avoid confusions with the notation for impact times $t_k$. 
5.3. MOREAU’S SWEEPING PROCESS

generated by the vectors $\nabla_q f_i(q)$, $i \in \mathcal{J}(q)$ \(^8\). $N(q) = \{0\}$ if $\mathcal{J}(q) = \emptyset$, i.e. if $q(t) \in \text{Int} \Phi$.

In the sweeping process formulation, the unknown will not be the position $q$, but its derivative, i.e. the velocities. More exactly, the unknown will be denoted as a time function $u$ such that

$$q(t) = q(\tau_0) + \int_{\tau_0}^{t} u(\tau) d\tau$$

(5.27)

Such a $u$ is evidently assumed to be Lebesgue-integrable, and it will be supposed for the moment locally absolutely continuous. The following propositions are in order:

**Proposition 5.1 ([381])** If $q(t) \in \Phi$ for every $t \geq \tau_0$, then

$$u(t^+) \in V(q(t)) \text{ and } u(t^-) \in -V(q(t))$$

(5.28)

Note that if $q(t)$ is in the interior of $\Phi$, this simply reduces to both right and left velocities to be in $\mathbb{R}^n$. Also for a smooth motion, $u(t)$ is continuous so that its left and right limits are the same; hence $u(t) \in V(q(t)) \cap -V(q(t))$, which is the linear subspace of $\mathbb{R}^n$ orthogonal to $N(q)$ (Hence the whole of $\mathbb{R}^n$ if $q \in \text{Int}(\Phi)$). In case of a single constraint, this set equals $T(q(t))$ in (5.25) if $f(q) = 0$, and otherwise the whole of $\mathbb{R}^n$.

Now if the boundary of $\Phi$ is attained at $t_k$, then necessarily $u(t_k^+)\nabla_q f_i(q(t_k)) \leq 0$ and $u(t_k^-)\nabla_q f_i(q(t_k)) \geq 0$. Roughly, the system must have attained some constraint $f_i(q) = 0$ and must either leave it or remain on it.

**Proposition 5.2 ([381])** Let $q(t)$ and $u(t)$ be associated as in (5.27). Suppose $q(\tau_0) \in q_S$, and that $u(t) \in V(q(t))$ Lebesgue-almost everywhere. Then $q(t) \in \Phi$ for all $t \geq \tau_0$.

**Remark 5.3** Notice that we do not care about the possible local existence problems, and we set for simplicity relationships on the whole of $\mathbb{R}^+$. In all the definitions and propositions, one may replace $\forall t \geq \tau_0$ by $\forall t \in I$, for some interval $I$. Our goal here is to provide the readers with an insight on this particular dynamical formulation of systems with unilateral constraints. It is clear that those who want to go in the deepest details concerning Moreau’s process are invited to consult the relevant sources we mentioned above. All proofs can be found therein.

Let us now consider the Lagrange equations of the system. We have already seen that the total reaction $P \in \mathbb{R}^n$ must be along the surface Euclidean normal for the case of one constraint, which generalizes to

$$P \in N(q)$$

(5.29)

\(^8\)This is true if $\Phi$ is convex around $q$. 
for several constraints. It is equivalent (see definition 5.2) to write 
\[ P = \sum_{i \in \mathcal{J}(q)} \lambda_i \nabla_q f_i(q), \]
with \( \lambda_i \geq 0 \). Hence the Lagrange equations can be written as
\[ -M(q)\ddot{q} + Q(t, q, \dot{q}) \in N(q) \quad (5.30) \]
that is a second order differential inclusion. We write \( Q(t, q, \dot{q}) \) to shorten the notations for Coriolis, centrifugal, gravity and bounded torques, see example 1.3. (5.30) is for the moment simply a rewriting of classical dynamical equations. Then a smooth motion agrees with the mechanical conditions stated (system inside the domain \( \Phi \), reaction in the normal cone to \( \Phi \)) if and only if (5.30) is satisfied and \( q(t) \in \Phi \) for all \( t \geq t_0 \). It is possible to show that every solution of the inclusion (5.30) in fact satisfies a stronger inclusion:

**Proposition 5.3 ([381])** A smooth motion with initial condition \( q(t_0) \in \Phi \), is a solution of (5.30) and verifies \( q(t) \in \Phi \) for all \( t \geq t_0 \), if and only if the velocity function associated to \( q \) in (5.27) satisfies Lebesgue almost everywhere the differential inclusion
\[ -M(q)\dot{q} + Q(t, q, u) \in \partial \psi_V(q)(u) \quad (5.31) \]
where the subdifferential is defined in definition D.4.

\[ \nabla \]

Recall that at this stage, \( u \) is locally absolutely continuous. For instance if \( m = 1 \), then \( N_{V(q(t))}(u(t^+)) \) is the outward normal halfline spanned by \( \nabla_q f(q) \). Some examples are depicted in appendix D.

**Nonsmooth motions** To deal with possible collisions, one needs to enlarge the space of functions \( u \) to that of local bounded variation functions (Actually, the hope is for the moment that the considered evolution problems possess solutions in this space). The trick is to replace Lagrange equations which are equality of functions by equality of measures. As we have discussed in chapter one, functions in \( RCLBV \) possess derivatives which can be identified with Stieltjes measures [477], and are a natural and convenient setting for the study of measure differential equations. If one associates to such \( u \) the measure \( du \) (called the differential measure of \( u \) in [381], and noted as \( Du \) in [477]), then one has that for any compact interval \( [t_1, t_2] \) on which \( u \) exists, \( \int_{t_1}^{t_2} du = u(t^+_2) - u(t^-_1) \). In particular, when \( t_1 = t_2 \), if \( u \) is discontinuous, then \( du \) possesses an atom at this point, i.e. it is a Dirac distribution (see appendices B and C for more details). Now using (5.27) the Lagrange equations of the system can be written as
\[ M(q)du + Q(t, q, u)dt = Pdt \quad (5.32) \]
which is an equality of measures. \( dt \) is simply the Lebesgue measure. All time functions possess the required smoothness so that the products in (5.32) are well defined. Clearly the term \( M(q(t))du \) in the left-hand-side of (5.32) is meaningful only because \( q \) is continuous, as we have previously respectively noticed and proved, see claim 1.1 and example 1.3. This way of writing the dynamics makes sense
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when $u \in RCLBV$. Also one can replace the right-hand-side of (5.32) by some real measures $dP$, which represent the total impulsion exerted on the system. This allows to encompass impulsive forces and torques. As we have seen in chapter one, such terms are in fact necessary when a discontinuity in the velocity occurs (see claim 1.1, example 1.3 and the section on variational calculus). Thus the right-hand-side of (5.32) can be written as $dP = F(t, q, u)dt + dR$, where $F(t, q, u)$ are the bounded generalized forces, and $dR$ are the contact impulsions (see chapter 1, section 1.1 and appendix B for the terminology associated to reactions at the contact point). Clearly in most applications

$$dR = \sum_{k=0}^{+\infty} p_k \delta_{t_k} dt$$

where the times $t_k$ correspond to the instants when $u$ is discontinuous. Since $u \in RCLBV$, the set of such times is countable (we shall use this important property of functions of bounded variation when we deal with stability of controlled systems with unilateral constraints in chapter 8). Hence the general form of $dR$ in (5.33)

With this material in mind, one easily deduces that the evolution problem at the times of discontinuity in $u$ can be written as

$$M(q)du - Q(t, q, u)dt = dR$$

Remark 5.4 Let us note that for the moment we have not made big advances with respect to what has been exposed in chapter one, see for instance example 1.3. It is indeed that the understanding of Moreau's process implies some necessary preliminaries, which cannot be avoided if one wants to be able to appreciate its powerfulness.

Consider smooth motions. From proposition 5.3 it follows that for Lebesgue-almost every $t$, one has

$$P(t) \in \partial \psi_{V(q)}(u)$$

This secures that $q(t) \in \Phi$ for all $t \geq \tau_0$, see propositions 5.2 and 5.3, and equation (5.30). One also has that $u \in V(q(t))$ for every $t$. Now at the discontinuities of $u$ one must choose how to replace $u(t)$ in the right-hand-side of (5.35). The following definition is then proposed:

Definition 5.3 (Soft shocks [381]) The set of unilateral constraints $f(q) \leq 0$ is said to be frictionless and soft if the total contact impulsion admits a representation $dR = R_{\mu} d\mu$, where $\mu$ denotes a nonnegative real measure, and $R_{\mu}$ is locally integrable (with respect to $d\mu$), such that for every $t$

$$- R_{\mu}(t) \in \partial \psi_{V(q)}(u(t^+))$$

$^9$Although as recalled in appendix C the derivative of a function $RCLBV$ is the sum of three main terms (see remark C.2), the singular nonatomic part of the derivative has mainly a theoretical interest, and it is difficult to exhibit a concrete example where it is not zero. Hence the general form of the contact impulsion in (5.33).
Thus, one replaces $u$ in the right-hand-side of (5.35) by its right limit $u(t^+)$, whereas the term $P(t)$ is replaced by the contact percussion $-R'_{\mu}(t)$, that is the density of the atom of the contact impulsion measure at the impact time $t_k$ (recall that a density of a measure is a function, see appendix B, definition B.3). Furthermore, $R'_{\mu}(t_k) = p_k$ where $p_k$ is given in (5.33). One feature of the sweeping process formulation is that the way one writes the dynamics is independent of the measure with respect to which the densities of the contact impulsion and of the acceleration (i.e. $du$) are expressed, provided this measure is nonnegative, [381] proposition 8.2.

This type of evolution problem is called a soft shock because it reduces when the surface of constraint has codimension one to the classical inelastic impact, i.e. a coefficient of restitution $e = 0$ (see example 5.3 below). We shall see in the following that the expression in (5.36) is equivalent to some more familiar impact dynamics expressions we have derived before.

**Remark 5.5** Notice that (5.36) is stronger than (5.29), because the latter is true for any impact process, whereas (5.36) implies a particular impact process. This is even more noticeable on the equivalent formulations of the sweeping process given below.

We are now in position to formulate the general sweeping process problem [366] [378]:

**Problem 5.1 (Frictionless sweeping process)** Find a $RCLBV$ function $u$ such that $u$ and the function $q$ defined by (5.27) satisfy the following

- $q(t_0) = q_0$
- $u(t_0) = u_0$
- $q(t) \in \Phi$ for all $t \geq t_0$
- $u(t) \in V(q(t))$ for all $t \geq t_0$
- $Q(t, q, u)dt - M(q(t))du \in N_{V(q(t))}(u(t))$

in the so-called sense of differential measures: there is a (non-unique) positive measure $\mu$ with respect to which the Lebesgue measure $dt$ and the Stieltjes measure $du$ both possess densities, respectively $t'_\mu = \frac{dt}{d\mu}$ and $u'_\mu = \frac{du}{d\mu}$ such that

$$Q(t, q, u)t'_\mu - M(q(t))u'_\mu \in N_{V(q(t))}(u(t))$$

(5.37)

$\mu$-almost everywhere.
Figure 5.8: Collision at a singularity (sweeping process).

See appendix B, definition B.3 for details about the notations for densities. In case of a percussion at \( t = t_k \), then (5.36) holds and the problem 5.1 becomes

\[
-M(q(t_k))u(t_k) \in N_{V(q(t_k))}(u(t_k^+))
\]  

(5.38)

(5.38) can be deduced noticing that the Lebesgue measure has no atoms. Indeed at the times of discontinuities, one has \( u(t_+^+) - u(t_-^-) = \sigma_u(t_k) = \nu_k t_k^+ \), and \( u(t_k^+) = 0 \). Hence multiplying both sides of (5.37) by \( u(t_k) \), one gets (5.38).

From problem 5.1 one might think that the post-impact velocity must be known to integrate pre-impact motion, but the following is true

**Proposition 5.4 ([381])** For any motion satisfying (5.37), one has

\[
u(t_+) = \text{prox} \left( u(t^-), V(q) \right)
\]

where the proximation is understood in the sense of the kinetic metric.

\( \nabla \nabla \)

A sketch of the proof is as follows: let us consider the codimension one constraint case and \( q \in \partial \Phi \), so that \( V(q) \) does not reduce to the whole of \( \mathbb{R}^n \). Suppose further that \( u(t^-) \in \text{Int}(-V(q)) \), i.e. a shock occurs. One may also consider \( M(q) \) as the identity matrix since the system is analyzed at a constant \( q \). There are two possibilities: either \( u(t^+) \in \text{Int}(V(q)) \) or \( u(t^+) \in \partial V(q) \) (recall that \( u(t^-) \in -V(q) \)). In the first case, then \( \partial \psi_{V(q(t))}(u(t^+)) = \{0\} \) and (5.38) implies \( u(t^-) = u(t^+) \) which is impossible. Hence the second possibility is true, and one may easily visualize that (5.39) holds. On the other hand, if (5.39) is satisfied and \( u(t^-) \notin V(q) \), then \( \sigma_u(t) \) belongs to the normal direction to \( \partial \Phi \) at \( q \). The different situations are depicted on figure 5.9. The cases when \( q \in \text{Int}(\Phi) \) and/or \( u(t^-) \in \partial(-V(q)) \) can be analyzed similarly. Hence the velocity after the shock is the vector closest (in the kinetic metric distance sense so that coordinate invariance is guaranteed) to the velocity before the shock, inside \( V(q) \). If \( u(t^-) \in V(q) \) then \( u(t^+) = u(t^-) \) so that no impact occurs. Some examples are illustrated in figure 5.8, inspired from
The computation of proximal points is at the core of the discretization of the sweeping process for numerical integration. From the fact that $u(t^-) \in -V(q)$, see proposition 5.1, $u(t^+)$ lies on the boundary of $V(q)$, i.e. generalized dissipative impacts are treated that correspond to $e = 0$ in the one dimensional case.

The reader can recognize in (5.36) and (5.38) a generalized formulation of the impact dynamics we have derived in simple cases in chapter 1. (5.36) and (5.38) may also be used to understand where the apparently complex formulation of problem 5.1 comes from. In case when $M(q)$ is the identity, (in general locally around a given $q$, so that both the tangent space to the configuration space and its dual, the cotangent space are locally identified to $\mathbb{R}^n$ at each $q$, 10), (5.36) and (5.38) are equivalent to the following conditions

\[
\begin{align*}
    u(t_k^+) &= V(q(t_k)) \\
    -R'_\mu(t_k) &= N(q(t_k)) \\
    u(t_k^+)^T R'_\mu(t_k) &= 0 \\
    \sigma_u(t_k) &= R'_\mu(t_k)
\end{align*}
\]  

(5.40)

The equations in (5.40) possess the form of a complementarity problem. We shall come back in more details on such formulations later in this chapter. The equivalence between (5.40) and $-R'_\mu(t) \in \partial \psi_{V(q(t))}(u(t^+))$ is shown using the definitions of the various terms appearing in those formulas, and convex analysis tools.

10This is a manner of identifying the space of generalized forces -the cotangent space- to the linear tangent space of generalized velocities. Recall that the impact relationships hold for a constant position $q(t_k)$. Hence these identifications are legitimate. Since the formulation at an impact instant $t_k$ is related to a position $q(t_k)$, one may just consider that the inertia matrix is constant and write the dynamics in a basis orthonormal in the kinetic metric at $q(t_k)$. $M(q(t_k))$ is a so-called tangent Euclidean metric of $\partial \Psi$ at $q(t_k)$, tangent to the kinetic metric.
The equations in (5.40) are in turn equivalent to

\[
\begin{align*}
    u(t_+^k) &\in \text{prox} \left( V(q(t_k)), u(t_k^-) \right) \\
    -R'_\mu(t_k) &\in \text{prox} \left( N(q(t_k)), u(t_k^-) \right) \\
    u(t_+^k)^T R'_\mu(t_k) &= 0
\end{align*}
\] (5.41)

The equivalence between (5.40) and (5.41) can be shown via direct application the lemma of the two cones, see lemma D.1, by identifying \( x \) with \( u(t_+^k) \) and \( y \) with \(-R'_\mu\), recalling that \( V(q(t_k)) \) and \( N(q(t_k)) \) are a pair of mutually polar closed convex cones of the Euclidean space \( \mathbb{R}^n \) (recall that the inertia matrix is considered to be the identity matrix since we work at fixed \( q(t_k) \), so the kinetic metric is the Euclidean one).

**Example 5.3** In order to clarify this kind of formulation, let us consider the dynamics of a ball falling vertically on a soft rigid ground within this framework: the dynamics can be written as

\[
-m du + g \in \partial \psi_{V(x)} \left( u(t^+) \right)
\]

where \( x \) is the vertical coordinate of the ball, \( u \) equals \( \dot{x} \) almost everywhere, and \( \partial \psi_{V(x)} \left( u(t^+) \right) \) is simply the normal halfline to the contact point when there is contact, or the singleton \( \{0\} \) when there is no contact. Also \( V(x = 0) \) is the inward normal halfline to the constraint surface, and \( N(x = 0) = -V(x = 0) \), from which it follows that \( u(t^+) = \text{prox}(u(t^-), V(x = 0)) = 0 \): thus here the formulation corresponds to the case \( \epsilon = 0 \). The case of non purely dissipative percussions may be treated by replacing \( u(t^+) \) by a weighted mean \( u^\delta = \frac{1+\delta}{2} u(t_+) + \frac{1-\delta}{2} u(t^-) \) of \( u(t^+) \) and \( u(t^-) \) in the right hand side of (P) [381], so that \( \delta = 0 \) yields energy conservation, \( \delta = 1 \) is a soft impact, and \( u(t^+) + (1-\delta)u(t^-) = \text{prox}(u(t^-), V(q)) \). The example of two rigid bodies that collide is explained in [381] in the framework of the sweeping process formulation.

**Remark 5.6** The essence of the sweeping process is to express the effect of the unilateral constraints on the system's dynamics, through the definition of the tangent cone \( V(q) \) and using convex analysis tools, in terms of a relationship between the contact impulsion density \( R'_\mu \) and the velocity. The obtained dynamical equations form a Measure Differential Inclusion.

### 5.3.3 Existential results

In the foregoing subsection, we have presented the formulation of the dynamical equations, which are a special kind of differential inclusions. As one might expect, it is natural to study the existence of solutions to this evolution problem. The main contributions to this field can be found in the book of Monteiro-Marques [366].
is worth noting that despite of the differential inclusion formulation in problem 5.1, the solutions of the sweeping process are not reachable sets (see e.g. [588]) but time functions as in an ordinary differential equation.

Consider the general formulation at the beginning of this section. A cornerstone of the approach is the proof of existence and uniqueness of solutions of the sweeping process when $C(t)$ may possess different properties [366]: by using so-called Yosida's approximants (which may represent some spring-like effect [416]) and showing convergence in the filled-in graphs sense when $C(t)$ is a right-continuous function of bounded variation, by using the so-called catching-up algorithm to construct uniformly convergent approximants (in the sense of discretization of the solution [379], a nice feature of the sweeping process being its natural property of being discretized) when $C(t)$ is Hausdorff continuous and has nonempty interior, or when $C(t)$ is right lower semicontinuous with nonempty interior (Note that as shown in [379], in case of dynamics with unilateral constraints, nonemptiness of the interior of $V(q(t))$ has to be assumed: a simple counter-example is given in [379]).

The case of a unique constraint $m = 1$ is treated in [366], where the following is proved:

**Theorem 5.1 (Monteiro-Marques [366])** Assume the vector field $Q(t, q)$ in problem 5.1 is continuous and globally bounded, i.e. $||Q(t, q)|| < M$ for some constant $M > 0$ and all $t \geq \tau_0$ and all $q \in \mathbb{R}^n$. Let us take also $M(q)$ the identity matrix. Let $q(\tau_0) = q_0 \in \Phi$ and $u(\tau_0) = u_0 \in V(q_0)$ be the initial data. Then there exist $\delta > 0$ and $T' > 0$ such that

$$\text{Int} \left( \cap_{||q-q_0|| \leq \delta} V(q) \right) \neq \emptyset$$

(5.43)

and $T' = \min \left\{ T, \frac{\delta}{M} \right\}$, $M' = ||u_0|| + 2TM$. On the interval $[\tau_0, T']$, problem 5.1 has at least one solution $q(t)$ with right-velocity $u$ that satisfies

$$||u(t)|| \leq ||u_0|| + Mt$$

(5.44)

so that

$$||q(t)|| \leq ||q_0|| + ||u_0||t + \frac{1}{2}Mt^2$$

(5.45)

Theorem 5.1 assures local existence, i.e. for some $T'$ strictly positive. Global existence can also be proved, i.e. a solution exists on $[\tau_0, T]$ for $T \geq \tau_0$, arbitrary.

Uniqueness is not discussed in [366], but it is pointed out that nonuniqueness should prevail, see chapter 2. However it seems that the only counter-example to nonuniqueness known in the mathematical literature [69] [476] (that is however to be considered more as a pathological case than as a deep problem, see subsection 2.2.3) does not apply to the sweeping process with soft constraints [368]. Mere existence of a solution to problem 5.1 seems quite difficult to prove when there are several constraints [366] §4.3. Note that for systems with unilateral constraints,
the word "several constraints" means in fact that several hypersurfaces are attained simultaneously. If their intersection is not smooth (if it is smooth, then considering several constraints is meaningless, since the normal direction is well-defined), then the normal direction is not well-defined. This is in general a singularity in multibody-multiconstraint systems. The sweeping process proposes a rule to calculate the postimpact velocity, but the existential problem is not obvious. We shall discuss in more detail about such multiple impacts in chapter 6.

The case of externally induced collisions with \textit{a priori} knowledge of the activation multifunction which describes the contacts of a solution with the constraints surface is studied in [366] §4.3; continuous dependence of the solution in the initial conditions fails in general: this is easily understandable from the following example [366] p.129: imagine in dimension 2 a convex constraint surface whose boundary is composed of two straight lines that intersect at \( P \) (see figure 5.10) The sweeping process formulation implies that if the singularity is attained directly, then the system remains at rest at \( x = y = 0 \) for all future times. If the system first strikes one of the surfaces of the multiple constraint, then it moves along this surface up to the origin, and starts moving along the other surface. This is an example of constraint with nonsmooth boundary. As we shall see in chapter 6, this kind of singularity occurs also in higher dimension spaces when we consider Lagrangian systems in configuration spaces. Then a problem like the rocking block can be set into the framework depicted in figure 5.10. See also subsection 6.5.8 about conditions guaranteeing continuity of the trajectories in initial data for multiple impacts.

Notice that these conclusions are however not in contradiction with theorem 9 of [477] (see subsection 1.2) on continuity of solutions of measure differential equation in state initial conditions: both problems are merely of different nature, since a change in the initial position in the example modifies the contact percussion in the dynamical equation.

Clearly properties of the solutions depend heavily on the problem's data such as the constraints smoothness, dimension and shape, the external forces acting on the system, and the initial conditions. For instance [47] shows that arbitrarily small
deformations of a closed circular domain within which a particle rebounds (the so-called billiard problem), implies drastic changes of the trajectories global behaviour. We also refer the reader to chapter 6 for further comments on this example, and on the problems related with multiple constraints.

**Remark 5.7** In the case of frictionless constraints with nonzero restitution, the sweeping process formulation and the one in [416] are equivalent [368]. However the counter-example on uniqueness given in [69] and [476] does not seem amenable for the inelastic sweeping process in problem 5.1. Apparently the elastic nature (0 < ε < 1) of the considered problems in [69] and [476] plays an important role in the existence of the counter-example, related to time-reversibility \(^{11}\) of such problems [368].

### 5.3.4 Shocks with friction

It is note worthy that the sweeping process may be applied to oblique collisions with friction as well [366] [381].

For the sake of simplicity, let us follow [380] and consider the contact at A between two bodies \(S_1\) and \(S_0\), \(S_0\) being fixed. Let \(V_{A_1}\) denote the velocity of the point belonging to \(S_1\) that coincides with A, expressed in a 3-dimensional frame. Then \(V_{A_1}^T n = 0\), if \(n \in \mathbb{R}^3\) is the common normal vector at A of \(S_1\) and \(S_0\) (this is equivalent to \(v_{2,n} = 0\) in the notations chosen in chapter 4, section 4.1. Here we fix body 1 and call it \(S_0\).) As we have seen in chapter 4, section 4.1, \(V_{A_1}\) is a linear function of \(q_1\) (the generalized velocity of \(S_1\)), i.e. \(V_{A_1}^T = E_3 M_1 q_1\), \(E_3 = [I_3; 0_{3 \times 3}]\), see (4.11). There is at A an interaction force exerted by \(S_0\) on \(S_1\), denoted as\(^{12}\)

\[
F_1 = F_1,t + F_1,n n
\]  
\[(5.46)\]

(eventual torques are neglected): \(F_1,t \in \mathbb{R}^3\) is the component of the interaction force in the tangent plane \(T\) between \(S_1\) and \(S_0\) at A. One has \(Q_1 = M_1^T E_3^T F_1\), where \(Q_1\) is the generalized force on \(S_1\) (see (4.4)).

Coulomb's friction cone \(C\) is classically defined in the 3-dimensional Euclidean space, at A. Coulomb's law states in particular that \(F_1 \in C\). This is equivalent to \(Q_1 \in C_q\), where \(C_q\) is the cone containing \(M_1^T E_3^T n\) (i.e. the image of the 3-dimensional "real-world" cone \(C\) in the configuration space of the coordinates \(q_1\)). The condition \(F_1 \in C\) is equivalent to

\[
F_1,t \in D(F_1,n)
\]  
\[(5.47)\]

\(^{11}\)In case \(ε = 0\), it is not possible to solve the inverse the problem since the past velocities are "erased" by the restitution law, contrarily to the cases \(ε > 1\) where the pre-impact velocities can be calculated from the post-impact ones. This is related to Carnot's theorem, which states that the imposition of persistent constraints is a nonreversible phenomenon [376].

\(^{12}\)The tangential component in (5.46) is denoted in boldface to emphasize that it is a vector of the 3-dimensional Euclidean space, whereas \(F_1,n\) is a scalar.
where
\[ D(F_{1,n}) = \begin{cases} F_{1,n}D_1 & \text{if } F_{1,n} \geq 0 \\ \emptyset & \text{otherwise} \end{cases} \tag{5.48} \]

where \( D_1 \) is the orthogonal projection on the tangent plane \( T \) between both bodies at \( A \), of the plane section of \( C \). More clearly, if the Coulomb’s law is chosen isotropic, then \( D_1 \) is a disc of radius equal to the friction coefficient, centered at \( A \). Then from (5.46) together with (5.47) Coulomb’s friction law can be expressed as:
\[ -V_{A_1} \in \partial \psi_{D(F_{1,n})}(F_{1,t}) \tag{5.49} \]
which can equivalently be written as
\[ -V_{A_1} \in \text{proj}_T \partial \psi_C(F_1) \tag{5.50} \]
(which means that the velocity belongs to the projection on \( T \) of the outward normal cone to \( C \) at \( F_1 \), see figure 5.3.4) or as
\[ F_{1,t} \in \partial \varphi(-\dot{q}_1) \tag{5.51} \]
where \( \varphi = \mu \| \cdot \| \), with \( \mu \) the friction coefficient. It is worth noting that the formulation in (5.50) is valid for any reaction \( \lambda F_1 \), \( \lambda > 0 \): only the direction of \( F_1 \) is involved [366] p.79. Hence the decomposition of the reaction into tangential and normal components is abandoned. These features are shown to be essential to formulate the dynamics in terms of differential measures, see problem 5.2.

The proof is given in [380] [381]. For instance, the equivalence between (5.49) and (5.51) can be shown using the fact that the dual function of \( \| \cdot \| \) is the indicator function of the ball \( \| y \| \leq 1 \) (see appendix D), and that the velocity and the interaction force belong to dual spaces (that can, anyway, both be identified with \( \mathbb{R}^3 \)). In other words, in (5.51) \( \varphi^*(y) = \psi_{\gamma\|y\|\leq1}(y) \), or (see theorem D.1) \( \psi_{\gamma\|y\|\leq1}^*(u) = \varphi(u) \), so that \( \partial \varphi(u) = \partial \psi_{\gamma\|y\|\leq1}^*(u) \) is the subdifferential of the support function of the convex set \( \gamma\|y\| \leq 1 \) (Coulomb’s law is expressed under this form in [299] [300]). Note that \( u \) and \( y \) belong to spaces respectively dual one to each other: if \( u \) is a velocity then \( y \) is a force.

Hence the dry friction phenomenon can also be put into a sweeping process formulation, using convex analysis language. It is note worthy that since Moreau’s formulation includes eventual tangential shocks (that we shall see in more details in subsection 5.4.2), (5.50) is also valid for impulsive interaction force and discontinuous velocity. It is assumed [381] §12, that the density of the contact impulsion satisfies the same kind of relations with the velocity as forces do.

A general formulation of the sweeping process with dry friction for a \( n \)-degree-of-freedom system with a single unilateral constraint is therefore given by:

**Problem 5.2 (Sweeping process with friction [366])** Find a \( RCLBV \) function \( u \) such that \( u \) and the function \( q \) defined by (5.27) satisfy the following
• \( q(\tau_0) = q_0 \)
• \( u(\tau_0) = u_0 \)
• \( q(t) \in \Phi \text{ for all } t \geq \tau_0 \)
• \( u(t) \in V(q(t)) \text{ for all } t \geq \tau_0 \)

and the following implications are true \( \mu \)-almost everywhere:

• \( f(q(t)) < 0 \Rightarrow R'_\mu(t) = 0 \)
• \( f(q(t)) = 0 \text{ and } u(t)^T \nabla_q f(q(t)) < 0 \Rightarrow R'_\mu(t) = 0 \)
• \( f(q(t)) = 0 \text{ and } u(t)^T \nabla_q f(q(t)) = 0 \Rightarrow -u(t) \in \text{proj}_{\mathcal{T}(q(t))} \mathcal{N}_{C(q(t))} (R'_\mu(t)) \)

where \( \mu \) is any positive measure such that \( R'_\mu \) can be defined with (5.34) and \( R'_\mu = \frac{d R}{d \mu} \).

\( \nabla \nabla \)

\( R'_\mu \) denotes as above the interaction force at the contact point. From (5.49) (5.50) (5.51) it is clear that the right-hand-side of the last implication can be written differently. \( T(q(t)) \) denotes the tangent hyperplane at the contact point and \( C(q(t)) \) is the generalized friction cone. The first implication means that when the bodies are not in contact, then the interaction force is zero. The second implication means that if there is contact, but the velocity points inwards the domain \( \Phi \), the interaction force is zero also (this is a kind of grazing point). The third statement means that when the velocity is tangential to the constraint surface, then its opposite belongs to the tangent hypersurface \( \mathcal{T}(q(t)) \) at the contact point \( q(t) \), and is in this hypersurface the point the closest to the outward normal cone of \( C(q(t)) \) at \( R'_\mu \). For instance consider the two-dimensional case on figure 5.3.4. In particular when \( R'(\mu) \in \text{Int}(C(q(t))) \), then the tangent cone to \( C(q(t)) \) at \( R'_\mu \) is the whole of \( \mathbb{R}^2 \), so that the normal cone reduces to the zero 2-vector. Consequently one gets \( -u(t) \in \text{proj}_{\mathcal{T}(q(t))} \{0\} = 0 \), and one retrieves that when the interaction force lies (strictly) inside the friction cone, the tangential velocity is zero. When the reaction lies on the boundary of the friction cone, then indeed (5.50) holds. Hence one still formulates Coulomb's friction law via convex analysis tools. Notice that this formulation encompasses all possible motions, with or without shocks. When there is a collision, the last condition in problem 5.2 is equivalent to

\[
 u(t) = \text{prox} \left( 0, [u(t^-) + C(q(t))] \cap T(q(t)) \right) \quad (5.52)
\]

Different situations are depicted in figure 5.12. Let us recall that the proximation is understood in the kinetic metric.

It is assumed in [366] that the real-world isotropic friction law is transported into a generalized isotropic friction law in the configuration space. In other words,
5.3. MOREAU'S SWEEPING PROCESS

![Sweeping process with friction](image1.png)

Figure 5.11: Sweeping process with friction.

![Sweeping process with friction (collision case)](image2.png)

Figure 5.12: Sweeping process with friction (collision case).

the friction cone $C(q)$ in the configuration space is revolving about $\nabla_q f(q)$, with vertex at the contact point. Global existence of a solution to the problem 5.2 is proved in [366]. We shall retrieve the sweeping process formulation when we deal with tangential impacts in subsection 5.4.2.

Additional comments and studies

Some related existential problems have been presented in [298] [299] [300]: the system consists of a particle moving on a rigid surface with Coulomb's friction, and acted upon by percussional effects (i.e. the external force acting on the particle has the general form $F = F_\text{ac}dt + F_\mu d\mu$, where $d\mu$ is a series of Dirac measures, whereas $dt$ is the Lebesgue's measure.

The sweeping process and the works in [187] [220] [222] aim at solving the same kind of problem, but with completely different methods. A critical comparison via simulations of these studies has not yet been done and would be interesting. The study of controllers for systems like walking machines or any mechanical system submitted to rigid body collisions will necessarily use such tools (which are in fact developed for this aim [220] [462]), and the available softwares devoted to nonlinear control systems (Simnon, Matlab) are not well suited. Note that two levels are to be distinguished: the basic problem of integrating a differential equation with a unilateral constraint (how to decide when the constraint is attained, which effects this may induce on the rest of the motion), and the postimpact motion of a complex mechanism where contacts may break (the first step is necessary, but not sufficient).
This problem has for instance led certain researchers in the field of robotics [359] to work with compliant models of environments. As we remarked previously, this may not always be a good procedure due to possible numerical problems [437]. Let us add that in parallel with these works, new friction models are studied [15] independently of any percussion considerations: one may again wonder what could be brought by these models in the impact theory. The state-of-the-art in the literature on impact (or nonsmooth) dynamics shows that this is still a widely open problem. For instance, a nice feature of some models (see [15] is that they are dynamical, i.e. the friction force and the relative velocity of the bodies are related through some differential equation. One might use such models to study convergence of a "totally compliant" problem of percussion towards a rigid one, which seems a reasonable way to understand what may happen in the rigid case. It is clear however that from an experimentalist point of view, percussions involve local phenomena which certainly imply to reconsider models developed for other purposes.

Existence of $C^1$ solutions in a one-degree-of-freedom system with friction

Note that the sweeping process formulation encompasses shock phenomena, but may also be used when no shocks occur, i.e. for tangential motions. In [367], Monteiro-Marques studies a one-degree-of-freedom system described by a differential inclusion:

$$M(q) \ddot{q} + F(t, q, \dot{q}) \in \Gamma(-\dot{q})$$

where $q$ is the system’s coordinate, and $\Gamma(u) = -a$ if $u < 0$, $\Gamma(u) = b$ if $u > 0$, $\Gamma(0) = [-a, b]$ represents the friction force. Equivalently one may write an ODE with discontinuous right-hand-side as:

$$M(q) \ddot{q} + F(t, q, \dot{q}) = \text{proj}_{\Gamma(-\dot{q})} F(t, q, \dot{q})$$

Denoting $\mathcal{F} \triangleq M(q) \ddot{q} + F(t, q, \dot{q})$, this can also be rewritten as:

$$\mathcal{F} \in \partial \psi_{[-a,b]}(-\dot{q})$$

or

$$-\dot{q} \in \partial \psi_{[-a,b]}(\mathcal{F})$$

where $\psi_{[-a,b]}$ is the indicatrix function of $[-a, b]$, and $\psi^*_{[-a,b]}$ is its conjugate (or dual) function, see appendix D for details. The basic assumptions are $F(\cdot,\cdot,\cdot) < +\infty$, $M(\cdot) \geq m > 0$, $M(\cdot)$ and $F(\cdot,\cdot,\cdot)$ are Lipschitz continuous. It is shown in [367] that $q$ is a solution of these differential inclusions if $q$ is a solution of the following problem $\mathcal{P}$:

$$q(0) = q_0, \dot{q}(0) = \dot{q}_0$$

$$\forall \varphi \in L^2, \varphi(t) \in [-a, b] \text{ almost everywhere, } \int_0^1 \dot{q}(t)[\varphi(t) - F(t)]dt \geq 0$$

$$\mathcal{F} \text{ almost everywhere } \in [-a, b]$$
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Then it is proved in [367] via a discretization-like procedure that under the above assumptions, $P$ possesses a global solution, which in turn satisfies the above differential inclusions almost-everywhere \(^1\). This solution is such that $q$ and $\dot{q}$ are continuous, whereas $\ddot{q}$ is of local bounded variation. The results are extended to $n$-degree-of-freedom systems, with the assumption that the normal component of the interaction force is known and constant (which is the same as assuming a persistent contact, disregarding the normal displacement) \(^2\).

5.4 Complementarity formulations

In the foregoing section, we described the sweeping process formulation that yields a (mathematically, mechanically and numerically) coherent and synthetic formulation generalized inelastic collisions, with or without friction. We have also highlighted the fact that this is an evolution problem in which the velocities are the unknowns. In fact, it belongs to the class of complementarity formulations, where the complementarity is defined between the velocity and the percussion, see (5.40). We already saw that the extension of Gauss’ principle (see chapter 3, section 3.1) uses complementarity conditions between the Lagrange multipliers and the second derivative of the constraints. The goal of this section is to provide the reader with more informations on complementarity formulations, which different forms they may take and which problems are related to them.

Consider the two body problem as in section 4.1. Let us construct the two points $A_1$ and $A_2$ such that $||A_1A_2|| = \min_{P_1 \in B_1, P_2 \in B_2} ||P_1P_2||$. In other words, $A_1$ and $A_2$ are the two points of bodies $B_1$ and $B_2$ respectively which are the closest one from each other. Consider one of the local coordinate frames we have constructed in chapter 4, section 4.1, e.g. $(A, n_1, t_{11}, t_{12})$. Let $X_{A_2} = (x_{2,n}, x_{2,t_1}, x_{2,t_2})$ be the coordinate vector of the point $A_2$ of body 2 in this local frame, with $\dot{x}_{2,n} = v_{2,n}$, $\dot{x}_{2,t_1} = v_{2,t_1}$. Then the so-called Signorini’s conditions state that

$$\begin{cases}
F_{2,n} \geq 0 \\
x_{2,n} \geq 0 \\
F_{2,n}x_{2,n} = 0
\end{cases} \quad (5.58)$$

The last equality simply means that if $x_{2,n} > 0$ then $F_{2,n} = 0$ and conversely. It is clear that at the impact times, $F_{2,n}$ is impulsive, and one can replace it by the

\(^1\)This is a current way of doing in functional analysis: one first transforms the system by integration, and proves the existence of so-called weak solutions in Sobolev spaces to the transformed problem (called the variational formulation of the problem). Then one shows that these weak solutions are regular enough so that they are solutions of the original problem [71].

\(^2\)This in particular precludes the application of the result in [367] to the Painlevé’s example described in subsection 5.4.2 on tangential impacts.
normal percussion component \( p_{2,n} \) in (5.58). But notice that such contact equations do not require the calculation of the shock impulsion to integrate the motion. This can be seen on a simple example [437]:

**Example 5.4** Consider the dynamics of the bouncing ball, i.e. \( m \ddot{x} = F - mg \), where \( F \) is the contact force \( (F \geq 0) \), and with the constraint \( x \geq 0 \). Hence the pair of functions \( F \) and \( x \) must verify the set of equations (5.58) to represent the dynamics of the system. Let us numerically integrate the motion as follows:

\[
\begin{align*}
\dot{x}_{n+1} &= \dot{x}_n - gh + F_{n+1} \frac{h}{m} \\
\dot{F}_{n+1} &= 0, \quad x_{n+1} \geq 0, \quad x_{n+1} F_{n+1} = 0
\end{align*}
\]

where \( h \) denotes the step of integration. Then the third equation in (5.58) can be written as

\[
(x_n + h\dot{x}_{n+1}) F_{n+1} = 0 \tag{5.60}
\]

Using the first equation in (5.59), one sees that this equation possesses two solutions given by

\[
F_{n+1} = 0, \quad \dot{x}_{n+1} = \dot{x}_n - gh \tag{5.61}
\]

and

\[
F_{n+1} = -\frac{m}{h} \left( \frac{x_n}{h} + \dot{x}_n - gh \right), \quad \dot{x}_{n+1} = -\frac{x_n}{h} \tag{5.62}
\]

If the system is initialized with \( x > 0 \), one starts by integrating it classically using its smooth dynamics, see (5.61). If the value computed at step \( n + 1 \) verifies \( x_{n+1} > 0 \), then this was the right solution. If on the contrary \( x_{n+1} \leq 0 \), then \( F_{n+1} \) and \( \dot{x}_{n+1} \) have to be calculated according to (5.62). The Signorini conditions are therefore equivalent to an inelastic impact, since when contact is made \( F \) becomes nonzero and \( x = 0 \) for all future steps.

One can also formulate **Signorini in velocity** conditions, capable of representing elastic impacts: if \( v_{2,n}(t_n^+) = -ev_{2,n}(t_n^-) \), then defining \( \ddot{v}_{2,n} = v_{2,n}(t_n^+) + ev_{2,n}(t_n^-) \), the conditions are given by

\[
\begin{align*}
F_{2,n} &\geq 0 \\
\dot{v}_{2,n} &\geq 0 \\
F_{2,n} \dot{v}_{2,n} &= 0
\end{align*}
\]

In fact, complementarity formulations are not necessarily used to describe impact dynamics, but rather dynamics of systems with unilateral constraints (which do not necessarily yield collisions between the bodies in contact, but require more than bilateral constraints as the complementarity formulation shows). The basic
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Complementarity formulation in position can equivalently be rewritten in velocity or in acceleration forms as

\[
\begin{align*}
    v_{2,n} & \geq 0 & a_{2,n} & \geq 0 \\
    F_{2,n} & \geq 0 & F_{2,n} & \geq 0 \\
    v_{2,n}F_{2,n} & = 0 & a_{2,n}F_{2,n} & = 0
\end{align*}
\]  

(5.64)

where we still use the notations introduced in chapter 4, section 4.1 and recalled above to describe the motion of two bodies. Note that it is understood that, at the time \( t_0 \) when these quantities are evaluated, the two bodies are in contact, i.e. \( x_{2,n}(t_0) = 0 \). Hence \( v_{2,n} \geq 0 \) constrains the possible evolution of \( v_{2,n} \) at \( t_0^+ \), and similarly for the acceleration. As we shall see in section 5.4.2, the formulation involving accelerations possesses the advantage that it is possible to include directly the dynamical equations into the complementarity problem. It is noteworthy that if \( F_{2,n} = p_{2,n}\delta_{t_k} \), then \( a_{2,n} = \sigma_{v_{2,n}}(t_k)\delta_{t_k} \) and it makes no sense to write equalities like \( v_{2,n}F_{2,n} = 0 \) and \( a_{2,n}F_{2,n} = 0 \): indeed as we saw in chapter 1, the first one involves the product of \( \delta_{t_k} \) with a function discontinuous at \( t_k \), and the second one is the product \( \delta_{t_k} \delta_{t_k} \), none of them being defined in the theory of distributions.

Other possible complementarity formulations are [247]

\[
\begin{align*}
    F_{2,n} & \geq 0 & x_{2,n} & \geq 0 \\
    \forall p_n \geq 0, (p_n - F_{2,n})x_{2,n} & \geq 0 & \forall s_n \geq 0, (s_n - x_{2,n})F_{2,n} & \geq 0
\end{align*}
\]  

(5.65)

Let us now consider a \( n \)-degree-of-freedom Lagrangian system, with a unilateral constraint \( f(q) \geq 0 \). Using the transformation that we describe in chapter 6, section 6.2, it is possible to write a general form of LCP (Linear Complementarity Problem) as follows

\[
\begin{align*}
    \ddot{q}_{\text{norm}} & \geq 0 \\
    \dot{q}_{\text{norm}}^T \lambda & = 0 \\
    \lambda & \geq 0
\end{align*}
\]  

(5.66)

that can be rewritten using the dynamical equations (see (6.6))

\[
\begin{align*}
    E_1\ddot{q} + n_q^T [C(q, \dot{q}) \dot{q} + g(q)] + n_q^T u + \bar{p}_q & \geq 0 \\
    [E_1\ddot{q} + n_q^T [C(q, \dot{q}) \dot{q} + g(q)] + n_q^T u + \bar{p}_q]^T \bar{p}_q & \geq 0 \\
    \bar{p}_q & \geq 0
\end{align*}
\]  

(5.67)

Rapidly, \( q_{\text{norm}} = f(q) \) is the distance between the system (point of the configuration space) and the surface, and \( \bar{p}_q = \sqrt{\nabla_q f(q)^T M^{-1}(q) \nabla_q f(q)} \lambda + n_q^T \lambda_t \), where \( M(q) \) is
the system’s inertia matrix. \( \lambda \) is the Lagrange multiplier associated to the constraint \( f(q) = 0 \), and \( \lambda_i \) accounts for dry Coulomb friction (which means that (5.67) is rather a Nonlinear CP, due to the friction cone conditions).

It is noteworthy that \( \lambda \) in (5.67) is multiplied by a scalar. If the constraint was of codimension \( \geq 2 \), this would be a matrix. Then one also has to include into the formulation the fact that some constraints may be active at \( \tau_0 \) (i.e. \( f_i(q(\tau_0)) = 0 \)) while others may be passive \( (f_i(q(\tau_0)) > 0) \). When facing such a dynamical problem formulated through its L- or N-CP, one must guarantee if possible existence and uniqueness of a solution (i.e. one calculates the multipliers at step \( j \) using the dynamics and then integrates the equations to find the subsequent motion at step \( j + 1 \)). General results on existence and uniqueness of solutions to LCP’s have been derived in the literature [115]. In particular the following is true, which provides a sufficient condition for a LCP to possess a unique solution:

\textbf{Theorem 5.2 ([115])} Consider the LCP

\begin{equation}
\begin{aligned}
z &\geq 0 \\
r + Mz &\geq 0 \\
z^T(r + Mz) &= 0
\end{aligned}
\end{equation}

If \( M \) is a \( n \times n \) P-matrix, for every \( r \in \mathbb{R}^n \), the LCP \( (r, M) \) has a unique solution. This solution \( z \) is a piecewise linear function of \( r \), hence Lipschitz continuous.

A matrix is called a P-matrix if all its principal minors have positive determinants\(^\text{15}\). The different terms \( z, r \) and \( M \) can be easily identified in the formulations above. It is clear that the above conditions that guarantee existence and uniqueness of a solution will not always be satisfied (they are for frictionless constraints [375]), but Coulomb friction may introduce additional difficulties, see subsection 5.4.2. It may then become necessary to consider nonsmooth motions to solve the LCP.

\subsection*{5.4.1 Additional comments and studies}

Recently, Moreau [385] introduced a new restitution rule as follows. First, a \textit{contact law} is a relation between the interaction forces (or their impulsions) and velocities. A contact law for two bodies colliding is said to be \textit{complete} if

\begin{equation}
\begin{aligned}
f(q) < 0 &\Rightarrow p = 0 \\
f(q) \geq 0 &\Rightarrow n^Tv \geq 0 \\
n^Tv > 0 &\Rightarrow p = 0
\end{aligned}
\end{equation}

\(^\text{15}\)A semi-positive definite matrix is a P-matrix, but P-matrices are not necessarily symmetric.
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where the unilateral constraint is \( f(q) < 0, \) \( n \in \mathbb{R}^3 \) is the normal to the common tangent plane at the contact point, \( p \in \mathbb{R}^3 \) is the contact force, and \( V \in \mathbb{R}^3 \) is the relative velocity between the two bodies at the contact point. In connection with proposition 5.2, let us define the set

\[
\mathcal{V}_q = \begin{cases} 
\mathbb{R}^3 : n^T V \geq 0 & \text{if } f(q) \geq 0 \\
\mathbb{R}^3 & \text{if } f(q) < 0
\end{cases}
\]  

(5.70)

This is the corresponding set to the set \( \mathcal{V}(q) \) in definition 5.1, expressed not for generalized velocities but for "real-world" 3-dimensional velocities. Then one may formulate proposition 5.2 in terms of the velocity \( V \) and the set \( \mathcal{V}_q \) to express invariance of the admissible domain \( \Phi \).

It is also proposed to write the contact law as follows

\[
\begin{align*}
& f(q) < 0 \Rightarrow p = 0 \\
& f(q) \geq 0 \Rightarrow n^T q^a_r \geq 0 \\
& n^T q^a_r > 0 \Rightarrow p = 0
\end{align*}
\]  

(5.71)

where \( q^a_r \) is an average relative velocity given by \( q^a_n = \frac{e}{1+\epsilon} \dot{q}_n(t_k^-) + \frac{1}{1+\epsilon} \dot{q}_n(t_k^+) \) in the normal direction, \( q^a_t = \frac{\epsilon}{1+\epsilon} \dot{q}_t(t_k^-) + \frac{1}{1+\epsilon} \dot{q}_t(t_k^+) \) in the tangential direction. \( \epsilon \) and \( e_1 \in [0, 1] \), and classical Newton’s rule is retrieved in the case of central impact, with non-zero initial velocity. When the contact is previously established (lasting contacts), the formulation provides a solution different from Newton’s rule and is thus applicable to line contacts (for instance a rectangular block falling on a plane, and rotating around one of its corners, see chapter 6, subsection 6.4.1).

The general form of the complementarity conditions as in (5.67) can be found for instance in the work by Lotstedt [320], Baraff [32] and Moreau [375]. In particular it is shown that quadratic programming and LCP’s are closely related, in the sense that a solution to the LCP is a solution to the following quadratic programming problem, and vice-versa:

\[
\begin{align*}
\min_{\lambda \geq 0} & \quad \frac{1}{2} \lambda^T \nabla_q f(q)^T M^{-1}(q) \nabla_q f(q) \lambda + \lambda^T \left( \nabla_q f(q)^T M^{-1}(q) z + \frac{d}{dt} (\nabla_q f(q))^T q \right) \\
\end{align*}
\]  

(5.72)

with \( z = -C(q, \dot{q}) \dot{q} - g(q) + Q \), where those terms have been defined in example 1.3. The dual problem in quadratic programming is given by

\[
\begin{align*}
\min_{\lambda} & \quad \frac{1}{2} \lambda^T \nabla_q f(q)^T M^{-1}(q) \nabla_q f(q) \lambda \\
\end{align*}
\]  

(5.73)

subject to

\[
\nabla_q f(q)^T M^{-1}(q) \nabla_q f(q) \lambda + \nabla_q f(q)^T M^{-1}(q) z + \frac{d}{dt} (\nabla_q f(q))^T q \geq 0
\]  

(5.74)
CHAPTER 5. MULTICONSTRAINT NONSMOOTH DYNAMICS

Writing the interaction forces term $\nabla_q f(q)\lambda$ as a function of the acceleration (using the dynamical equations), one retrieves Gauss' least action principle (see section 3.1).

The complementarity formulations, such as Signorini's conditions, lend themselves very well to numerical investigations [437]. Trinkle et al [543] cleverly justify the use of rigid body systems in the field of robotics, graphics, virtual reality. In particular, this yields dynamical problems that can be formulated through Linear or Nonlinear (because of friction) Complementarity Problems. Hence numerical methods to solve such NCP's must be found to allow the computation of contact forces, so that motion can be calculated. It is important that the proposed methods yield "believable" results. Although the addition of compliance at contact points leads to problems with unique solutions [569], this is not in general a good choice [543], mainly because the compliance can influence and modify the motion (in which proportion?): hence the predicted motion may be far from the real one. One topic of research is thus the resolution of LCP's and NCP's with numerical algorithms as efficient as possible [543] [32] [33] [319] [320]. Let us finally note that complementarity formulations arise also in other variational problems with unilateral constraints, see e.g. [109] theorem 1, who considers a problem of Mayer, with constraints of the form $f(q, \dot{q}) \leq 0$.

5.4.2 Tangential impacts or impacts without collisions

Until now, we have studied impacts phenomena that occur between two or several bodies that collide. We have seen that due to the unilateral constraints imposed to the system, the velocities possess in general discontinuities at the impacts which are directly related to the percussion between the bodies. We have also noted that rigid body dynamics may contain indeterminancies, i.e. it may be impossible to choose among several outcomes predicted by the theory. Another problem related to rigid body assumption is that of inconsistency: roughly, there are configurations of the system such that there are no solution at all! More precisely, there are no bounded solutions, and the space within which solutions have to be defined and found must be augmented to discontinuous velocities and distributional interaction forces. For instance some sort of "impacts without collisions" can occur when friction is present. This is a phenomenon such that velocity jumps can occur with zero initial normal velocity, primarily due to Coulomb's friction. This phenomenon is related to Kilmister's principle of constraints [278], according to which a unilateral constraint must be verified with forces each time this is possible, and with impulses if and only if it is not possible with forces. Hence according to this principle, if one is able to exhibit dynamical situations for which a force cannot be found such that a constraint is verified, then the solution may be replaced by an impulse at the contact point. It is interesting to note that Newton's rule is in this case meaningless since $v_n(t^-) = 0$ and $v_n(t^+) \neq 0$. This is in fact natural since these jumps are not related to any collision. Although this phenomenon is a little outside the scope of the study of shock dynamics, we briefly describe it since it involves contact impulsions. Finally
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Figure 5.13: Painlevé's example.

It is a problem closely linked to the unilateral feature of the constraints, and to the complementarity formulations described above. Note that inconsistencies due to Coulomb's friction have been known for a long time [50] [248] [304] [410] [185] [40] [319] [41] [125] [126] [283] [365] [186] [44]: after a certain time, the velocity is no longer deterministic, and suffers from a discontinuity. Historically, inconsistencies due to Coulomb friction have been noticed first by Painlevé [410]. The possibility of solutions with velocity discontinuities has been first recognized by Lecornu [304].

An example

To illustrate this phenomenon, let us consider the example of a planar slender rod sliding on a rigid surface, with Coulomb's friction at the contact, as depicted in figure 5.13. We introduce two sets of generalized coordinates: \( q = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \) and \( z = \begin{pmatrix} x_a \\ y_a \\ \theta \end{pmatrix} \). In \( q \)-coordinates, the constraint is given by

\[
0 = y - l \sin(\theta) = 0 \tag{5.75}
\]

and by

\[
y_a = 0 \tag{5.76}
\]

in \( z \)-coordinates. The external actions on the rod are gravity plus the reaction at the contact point \( A \), considered for the moment as bounded forces, i.e. \( \begin{pmatrix} F_t \\ F_n \end{pmatrix} \), in the Cartesian "real-world" frame. The friction coefficient is \( \mu \geq 0 \). To simplify the notations we shall write (in case of sliding regime) \( F_t = \mu F_n \) instead of \( -\mu \text{ sgn}(x_a)F_n \) for the tangential reaction due to friction, i.e. we assume that \( x_a \) is negative before any discontinuity occurs. Also we consider here only dynamic friction, i.e. the reaction lies on the boundary of the friction cone. The Jacobian between \( \dot{q} \) and \( \dot{z} \)

\footnote{Sometimes called [350] Painlevé's example [411].}

\footnote{With our previous notations, we clearly have \( \dot{y}_a = v_{2,n} \), and \( \dot{x}_a = v_t \).}
is given by $J(\theta) = \begin{pmatrix} 1 & 0 & l \sin(\theta) \\ 0 & 1 & -l \cos(\theta) \\ 0 & 0 & 1 \end{pmatrix}$, i.e. $\dot{z} = J(\theta)\dot{q}$. Since there is no torque acting on the rod at point A, the dynamical equations in a sliding regime are given by

$$
\begin{aligned}
\ddot{x} &= \mu F_n \\
\ddot{y} &= -g + F_n \\
I\ddot{\theta} &= (\mu \sin(\theta) - \cos(\theta)) l F_n
\end{aligned}
$$

(5.77)

in $q$-coordinates, and by

$$
\begin{pmatrix} 1 & 0 & -l \sin(\theta) \\ 0 & 1 & l \cos(\theta) \\ -l \sin(\theta) & l \cos(\theta) & l^2 + I \end{pmatrix} \begin{pmatrix} \ddot{x}_a \\ \ddot{y}_a \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} \mu F_n \\ F_n \end{pmatrix}
$$

(5.78)

in $z$-coordinates. We have assumed for simplicity that $m = 1$. The inverse of the inertia matrix in $z$-coordinates is given by

$$
M_z^{-1} = \begin{pmatrix} 1 + \frac{l^2}{I} \sin^2(\theta) & -\frac{l^2}{I} \sin(\theta) \cos(\theta) & \frac{1}{I} \sin(\theta) \\ -\frac{l^2}{I} \sin(\theta) \cos(\theta) & 1 + \frac{l^2}{I} \cos^2(\theta) & -\frac{1}{I} \cos(\theta) \\ \frac{1}{I} \sin(\theta) & -\frac{1}{I} \cos(\theta) & \frac{1}{I} \end{pmatrix}
$$

(5.79)

from which it can be calculated that

$$
\ddot{x}_a = l\dot{\theta}^2 \cos(\theta) + \left( \mu + \frac{l^2}{I} \sin^2(\theta) - \frac{l^2}{I} \sin(\theta) \cos(\theta) \right) F_n
$$

(5.80)

$$
\ddot{y}_a = A(\theta)\dot{\theta}^2 + B(\theta, \mu) F_n - g
$$

(5.81)

with

$$
A(\theta) = l \sin(\theta)
$$

(5.82)

and

$$
B(\theta, \mu) = 1 + \frac{l^2}{I} \cos^2(\theta) - \mu \frac{l^2}{I} \cos(\theta) \sin(\theta)
$$

(5.83)

Now following [32] §5, let us recall that we are looking for a solution to the dynamical problem such that the following constraints are verified

$$
F_n \geq 0 \quad \dot{y}_a \geq 0 \quad F_n \dot{y}_a = 0
$$

(5.84)
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The first two constraints are necessary not to violate the constraint \( y_a \geq 0 \). The third one comes from the fact that if the bodies break contact, then the interaction between them vanishes, and the relative acceleration must be positive. Notice that this is true because we consider the bodies initially in contact, performing a relative tangential motion. Hence the initial conditions on relative normal position and velocity are \( y_a(0) = \dot{y}_a(0) = 0 \). It is clear why as soon as \( \dot{y}_a > 0 \) then bodies break contact and the interaction force must become zero. These constraints together with the dynamical equations form a Linear Complementarity Problem, or LCP. Compare the conditions in (5.84) with those in (5.58) and (5.63), noting that we now do not \textit{a priori} deal with collisions, but simply contact problems.

It is clear that the nicest dynamical situation occurs when the LCP possesses a unique solution, see e.g. a sufficient condition in theorem 5.2. Then \( F_n \) can be calculated from (5.81) with \( \ddot{y}_a = 0 \).

**Remark 5.8** As we saw in the foregoing subsection, LCP's are closely related to quadratic programming. Indeed the LCP in (5.84) can be rewritten as

\[
\ddot{y}_a = AF_n - b \geq 0 \quad F_n \geq 0 \quad F_n \dot{y}_a = 0 \tag{5.85}
\]

where \( A = e_2^T M_a e_2 \) if there is no friction, \( e_2^T = (0, 1, 0) \), and \( A = e_2^T M_s \begin{pmatrix} \mu \\ 1 \\ 0 \end{pmatrix} F_n \) if there is dynamic friction. \( b \) is a function of the remaining dynamical terms. It is apparent that for the frictionless case, \( A \) is a positive scalar (In more general situations with constraints of codimension larger than 1, this would be a positive definite matrix). This is called a \textit{PSD-LCP} in [32]. Then the LCP can be phrased as a quadratic programm as follows:

\[
\text{minimize}_{F_n} AF_n^2 - bF_n \quad \text{subject to} \quad \begin{cases} \quad AF_n \geq b \\ \quad F_n \geq 0 \end{cases} \tag{5.86}
\]

The function \( AF_n^2 - bF_n \) is zero for every \( F_n \) that satisfies the LCP.

The whole problem is to find whether the LCP possesses a solution, several solutions or no solution. Several solutions correspond to an \textit{undetermined} problem, something currently encountered in rigid body dynamics. No solution (in the space of bounded \( p_n \)) corresponds to an \textit{unconsistent} problem: the space of possible solutions has to be modified in order to render the LCP solvable. Let us now illustrate \textit{indeterminancy} and \textit{inconsistency}.

**Indeterminancy**

Let us assume [32] that the configuration at time \( t \) is such that (we drop the argument \( t \) for convenience) \( A(\theta)\dot{\theta}^2 - g = 1 \), and that \( \ddot{y}_a = 0 \) (i.e. there is no normal
relative initial velocity). Let us further assume that \( B(\theta, \mu) = -1 \) (this is possible with suitable choice of \( \mu > 0 \)). Then the LCP in (5.84) becomes

\[
\begin{align*}
-F_n + 1 & \geq 0 \\
F_n & \geq 0 \\
F_n [ -F_n + 1 ] & = 0
\end{align*}
\] (5.87)

There are two solutions to (5.87): the first one is \( F_n = 0 \) (no reaction at the contact point, the rod is grazing the constraint), and the second one is \( F_n = 1 \) (The two bodies do not break contact). It is a priori impossible to choose among these two solutions: rigid body model tells us that they are both likely to occur. Indeterminacies are currently met in multibody-multiconstraint systems, see section 5.2.

**Inconsistency**

Let us now assume [32] that the configuration is such that \( \dot{\theta} = 0 \). The rod is sliding horizontally. The LCP in (5.84) hence becomes

\[
\begin{align*}
B(\theta, \mu)F_n - g & \geq 0 \\
F_n & \geq 0 \\
F_n [ B(\theta, \mu)F_n - g ] & = 0
\end{align*}
\] (5.88)

For the first equation in (5.88) to be verified together with the second one, one must have \( B(\theta, \mu) > 0 \). If this condition is verified, then there exists a bounded \( F_n \geq 0 \) solution of the LCP. Otherwise, if \( \mu \) is such that \( B(\theta, \mu) < 0 \), no bounded positive \( F_n \) can assure that \( \ddot{y}_a \geq 0 \). In other words, any bounded positive reaction at the contact point will not be sufficient to prevent interpenetration of the two bodies. Intuitively, if one tries to simulate such a configuration using a compliant approximating problem (replacing the surface by a linear spring-damper so that \( F_n = -ky_a - f\dot{y}_a \)), then (5.81) will become

\[
\ddot{y}_a = -B(\theta, \mu)(ky_a - f\dot{y}_a) - g
\] (5.89)

which represents an unstable order two system. Hence \( y_a \) may have the tendency to grow without bounds, and the interaction force will do as well. Of course it may happen that the system evolves such that the "position feedback" represented by the interaction force becomes negative, hence stabilization. But if we let \( k \) grow without bound, we see that by continuity of \( B(\theta, \mu) \) as a function of \( \theta \), there will be a nonzero time interval during which the interaction force will grow unbounded as well, since \( B(\theta, \mu) < 0 \) on this interval so that \( y_a \) continues to decrease (rigorously, one should lead an analysis based for instance on singular perturbations to
investigate the dynamical behaviour when the stiffness is very high). Anyway, what is interesting and noteworthy is the fact that Coulomb friction creates this sort of positive feedback, which would be impossible without friction, see remark 5.9.

In summary, given $\mu$ and the system's physical parameters, an indeterminancy occurs for couples $(\theta, \dot{\theta})$ such that the LCP in (5.84) has more than one solution. An inconsistency occurs for couples $(\theta, \dot{\theta})$ such that the LCP has no solution.

**Remark 5.9** Notice that if $\mu = 0$, then $B(\theta, 0) = \frac{\mu^2}{I} \cos^2(\theta) + 1 > 0$. Hence the frictionless case is always consistent.

Let us introduce the problem another way. Assume the system is initialized in a sliding regime, so that the LCP possesses a unique solution on an interval $[0, t_0)$ with $t_0 > 0$. In particular on $[0, t_0)$ one has $F_n \geq 0$ and $\ddot{y}_a = 0$. Classically one calculates $F_n$ from (5.81) to get

$$F_n = \frac{1}{B(\theta, \mu)} (g - A(\theta)\dot{\theta}^2)$$

(5.90)

If it happens that at $t_0^-$, $F_n(t_0^-) = 0$ and that at $t_0^+$, $F_n(t_0^+) < 0$, then the rod detaches from the surface (indeed there is only one constraint, and in this simple case it is known that the sign of the interaction force is sufficient to decide whether contact remains or ceases [429]). Another possible outcome is that $F_n$ becomes unbounded, if the denominator in (5.90) tends towards zero while the numerator remains bounded away from zero. Now it may also occur that $\ddot{y}_a(t_0^-) < 0$ (18). This case is much more intricate, since it means that the dynamical conditions (recall we are in a sliding regime) yield penetration into the constraint. At the same time, it is expected intuitively that $F_n$ grows unbounded, since any small amount of penetration into a rigid body implies an infinitely large interaction force. Hence the two phenomena (unbounded $F_n$ and negative acceleration) must be closely related one to each other. Note further that if $F_n \to \infty$, then necessarily from (5.90) $B(\theta, \mu) \to 0$, and from (5.80) one has that $\ddot{x}_n \to \infty$. What to do then? Intuitively, if there is no solution in the space of bounded functions, one wonders if an impulsive behaviour creating velocity discontinuities would solve the LCP. In other words, by augmenting the space of admissible reaction to that of distributions (or measures), it is expected that the LCP will be solvable so that at $t_0^+$ one gets $F_n(t_0^+) \geq 0$ and $\ddot{y}_a(t_0^+) \geq 0$.

Now note the following: inconsistency in the rod example comes from the fact that the configuration is such that $B(\theta, \mu) < 0$ and $\dot{\theta} = 0$. Now it is clear from (5.81) that there is always a value of $\dot{\theta} \neq 0$ such that even if $B(\theta, \mu) < 0$, then $F_n \geq 0$ and $\ddot{y}_a \geq 0$ are solutions of the LCP (This is true at least in the first quadrant for $\theta$, since then $A(\theta) > 0$). This means that if the configuration is inconsistent at time $t_0$, there exist at least a jump of the velocity $\dot{\theta}$ such that $\ddot{y}_a(t_0^+)$ renders the LCP solvable.

---

18Numerically, all these phenomena appear from one step of integration to the next one. It is then important to possess general rules that allow to continue the simulation process, and this is the goal of all the works that deal with algorithms to solve LCP's, see the foregoing section.
on an interval \((t_0, t_0 + \delta)\), for some \(\delta > 0\). But obviously from the developments on measure differential equations in chapter 1 such a jump must be accompanied by an impulsive reaction at the contact point: this is an *impact without collision*, or IW/OC. Then \(F_n = p_n\delta t_k\) and \(F_t = p_t\delta t_k\) for some \(t_k\) \(^{19}\).

Now some questions arise:

- Should an IW/OC occur only when there is inconsistency?
- How can we compute the velocity jump?
- Is it unique?
- Most importantly, if a system is initialized in a sliding regime, will it attain an inconsistent configuration of its LCP or not?
- If such an IW/OC occurs, where does it originates from? What is the physical phenomenon that produces it?
- Is it necessary that an IW/OC occurs only when \(F_n\) grows unbounded in finite time, so that \(\ddot{x}_a\) does, whereas \(\ddot{y}_a\) has the tendancy to become negative? How are those phenomena related? \(^{20}\)

Kilmister's principle of constraints \[^{278}\] proposes a positive answer to the first question. Although this seems at first sight quite natural, some authors argue that there is no reason for such principle to be true \[^{32}\]. In fact this is supported by the analysis of approximating compliant problems, where due to some energy transfer between angular and linear normal directions, there may be a jump even without any friction. This example however requires more investigations, in particular the rigorous study of the limiting rigid problem. Concerning the second question, Baraff \[^{32}\] §8.1 proposes the following rules

**Baraff's rules for inconsistencies**

- Since inconsistency is caused by dynamic friction, the impulse must convert at least one of the contact points to static friction \[^{21}\].
- The contact impulse must be such that the bodies do not separate after the discontinuity.

\[^{19}\] We adopt here the notation \(t_k\) for an eventual time of tangential impact to remain consistent with the rest of the book.

\[^{20}\] In particular, note that if it is true that \(F_n\) must grow unbounded before any velocity jump, this makes the IW/OC phenomenon very different from the classical collisions. Indeed in classical shocks, the other forces need not at all be unbounded for a velocity discontinuity to occur. In a sense, the collision phenomenon is quite isolated from the rest of the dynamics (hence the atomic nature of the contact impulsion, see chapter 1).

\[^{21}\] It is shown in \[^{32}\] theorem 4, that 2-dimensional one-contact configurations with static friction are always consistent. This is the case for the Painlevé's example.
5.4. COMPLEMENTARITY FORMULATIONS

The first statement signifies in the sliding rod example that the jump in $\dot{\theta}$ must be such that the rod stops sliding, i.e. $\dot{x}_a(t_k^+) = 0$. The second statement implies $\dot{y}_a(t_k^+) = 0$.

Baraff's supports the definition of a new principle of constraints from the consideration of an approximating compliant problem. Of course rules based on conclusions drawn from a compliant problem are always subject to being possibly contradicted if one considers another approximating problem.

Remark 5.10 Note that without any additional rule, there may be an infinity of possible jumps such that the LCP is verified. Adding rules clearly is the same as adding constraints, and reduces the number of outcomes. However the rules should be such that they do not introduce new inconsistencies where there should not be.

Let us illustrate the computations on the rod example, with the numerical values chosen above, i.e. the inconsistent case where $\dot{\theta}(t_k^-) = 0$ and $B(\theta, \mu) = -1$, i.e. $\mu = \frac{2I + l^2 \cos^2(\theta)}{2 \cos(\theta) \sin(\theta)}$ from (5.83). From the fact that $\dot{y}_a(t_k^-) = 0 = -F_n(t_k^-) - g$, a jump in $\dot{\theta}$ is needed to render the LCP solvable. Let us recall the LCP, which can be written as follows after the impact:

$$\begin{cases} A(\theta)(\dot{\theta}(t_k^+))^2 - F_n - g \geq 0 \\ F_n \geq 0 \\ F_n[A(\theta)(\dot{\theta}(t_k^+))^2 - F_n - g] = 0 \end{cases} \quad (5.91)$$

Applying Baraff's first rule, we must have also $\dot{x}_a(t_k^+) = 0$. Notice further that whatever occurs, if there is a jump in the velocities at time $t_k$, then the following equations are verified at $t_k$:

$$M_x \begin{pmatrix} \sigma_{x_a} \\ \sigma_{y_a} \\ \sigma_\theta \end{pmatrix} = \begin{pmatrix} p_t \\ p_n \end{pmatrix} \quad (5.92)$$

where $p_n$ in (5.92) represents the normal impulse magnitude, and $p_t$ is the tangential impulse magnitude. Notice that we cannot assume now that $p_t = \mu p_n$. Indeed the contact percussion density, i.e; the percussion vector (recall definition 1.2) $P = p_n n + p_t t$ may lie either inside or on the boundary of the friction cone. Since the inconsistency has been shown assuming that dynamic friction occurs, it is necessary to suppose that $\dot{x}_a(t_k^-)$ is not zero, i.e. there is slipping at the contact point (Recall we have also made the assumption that $\dot{x}_a(t_k^-) < 0$ to simplify notations). Now we can use (5.92) to compute $\dot{x}_a(t_k^+)$ as:

$$\dot{x}_a(t_k^+) = \left(1 + \frac{l^2}{I} \sin^2(\theta)\right) p_t - \frac{l^2}{I} \sin(\theta) \cos(\theta) p_n + \dot{x}_a(t_k^-) \quad (5.93)$$
and \( \dot{y}_a(t_k^+) \) as

\[
\dot{y}_a(t_k^+) = \left( -\frac{1}{I} \sin(\theta) \cos(\theta) \right) p_t + \left( 1 + \frac{1}{I} \cos^2(\theta) \right) p_n \tag{5.94}
\]

and \( \dot{\theta}(t_k^+) \) as

\[
\dot{\theta}(t_k^+) = \frac{l}{I} (p_t \sin(\theta) - \cos(\theta) p_n) \dot{\theta}(t_k^-) \tag{5.95}
\]

These equations are completely independent of the preimpact process. They are verified if any discontinuity in the velocities occur. Now let us apply Baraff’s rule, i.e. let us set \( \dot{x}_a(t_k^+) = \ddot{x}_a(t_k^+) = 0 \). There are 3 unknowns to the problem \( (p_n, p_t \text{ and } \dot{\theta}(t_k^+)) \) and 3 equations. Let us note that since it is admitted that the system may escape from the inconsistent configuration by sticking, it is possible that the impulsion vector \( \left( \begin{array}{c} p_t \\ p_n \end{array} \right) \) lies in the interior of the friction cone, i.e. \( |p_t| < |p_n| \). From \( (5.93) \) \( (5.94) \) and \( (5.95) \) one calculates that

\[
p_n = \frac{-l^2}{I + l^2} \sin(\theta) \cos(\theta) \dot{x}_a(t_k^-) \tag{5.96}
\]

\[
p_t = -\frac{I + l^2 \cos^2(\theta)}{I + l^2} \dot{x}_a(t_k^-) \tag{5.97}
\]

\( \dot{\theta}(t_k^+) \) can be obtained from \( (5.95) \). Now notice that as long as \( \dot{y}_a(t_k^+) = \ddot{y}_a(t_k^-) = 0 \) and the percussion vector \( P \) is not zero, from \( (5.94) \) it follows that:

\[
\frac{p_t}{p_n} = \frac{I + l^2 \cos^2(\theta)}{l^2 \cos(\theta) \sin(\theta)} \geq \mu_c \tag{5.98}
\]

and that \( B(\theta, \mu_c) = 0 \). Depending on the relative value of \( \mu_c \) with respect to that of \( \mu \), the calculated impulsion may lie either inside, on the boundary or outside the friction cone \( C \). Clearly the last case is impossible, since it is assumed that the impulsion also satisfies Coulomb’s dry friction law. This means that if \( \mu_c > \mu \), then Baraff’s rules (in particular conversion of sliding into sticking) yield an impossible situation: for the system to be pulled outside of the inconsistent configuration, sliding must cease instantaneously, implying an impulse that does not satisfy Coulomb’s law. One notices however that \( p_t = p_n = 0 \) is always a solution of \( (5.94) \), which gives zero velocity jump. But then the system does not enter a sticking regime as required by Baraff’s rules when an inconsistency has been attained.

**Remark 5.11** Note that it is not claimed in [32] that the proposed method and rules always yield a unique solution to the \( LCP \). In particular since friction may render the \( LCP \) nonpositive definite, it may possess several or no solution. However it is claimed in [32] that Lemke’s algorithm [308] is suitable for the numerical treatment of such problems, since in any case it will provide a value that verifies the system’s constraints (here \( y_a \geq 0 \)). But then there may be detachment of the bodies,
and "true" collisions may occur in the future. Other numerical algorithms for solving LCP's have been proposed in [320] [319] where smoothness of the solution is a priori assumed. Trinkle et al [543] study the case with friction and propose a numerical method that guarantees calculation of a solution. Let us recall that Moreau's sweeping process discretization also leads to efficient numerical algorithms [380] capable of handling both smooth and possibly nondifferentiable motions.

Remark 5.12 From the expression of $\mu_c$ in (5.98), it can be calculated that $\mu_c(\theta) \geq \frac{4}{3}$. Therefore for all $\mu < \frac{4}{3}$, the LCP possesses a unique solution. $B(\theta, \mu)$ always remains bounded away from zero, so that $F_n$ cannot grow unbounded, and $\dot{x}_n$ neither. This is a particular case of more general results on the existence and uniqueness of smooth solutions of LCP's and NCP's for small enough friction coefficients, see for instance [350] [543].

Remark 5.13 Would another friction model help in removing the inconsistencies? It seems that as long as it incorporates a model of dynamic friction similar to the basic Coulomb's model, it will not be of any help.

Before introducing the algorithm proposed in [381] to cope with inconsistencies, let us recall the basic "invariant" facts about tangential impacts: if an IW/OC occurs at $t_k$, then the following conditions have to be satisfied:

- i) $T(t_k^+) \leq T(t_k^-)$.
- ii) $\dot{y}_a(t_k^+) = 0$.
- iii) $P \in C$, where $P \in \mathbb{R}^2$ is the percussion vector, and $C$ is the friction cone.
- iv) The LCP possesses a solution after the IW/OC.

Conditions i ii imply that the postimpact velocity $\dot{x}_a(t_k^+)$, $\dot{\theta}(t_k^+)$, lies in a compact subset of $\mathbb{R}^2$. Any particular rule chosen to compute the velocity jump must satisfy such requirements. We shall retrieve similar conditions in chapter 6 for the calculation of postimpact velocities, when the system strikes a constraint of codimension $\geq 2$. Condition iv provides a constraint on $\dot{\theta}(t_k^+)$ (22)

5.4.3 Lecornu's frictional catastrophes

This phenomenon has been named frictional catastrophes in [381], in the honour of L. Lecornu who was the first one to accept the possibility of velocity jumps to solve
inconsistencies. It appears in the sweeping-process formulation, when Coulomb’s friction is taken into account. Consider (5.92), and \( E_2 = [I_2 0_{2 \times 1}] \in \mathbb{R}^{2 \times 3} \). Then it follows that

\[
\begin{pmatrix}
\sigma x_a \\
\sigma y_a
\end{pmatrix} = H \begin{pmatrix} p_t \\ p_n \end{pmatrix} \triangleq HP
\]

(5.99)

where \( H = E_2 M_2^{-1} E_2^T \in \mathbb{R}^{2 \times 2} \) is full-rank. Moreau [381] chooses the following rule:

**Moreau’s rule**

- There exists \( \sigma \in \mathbb{R} \) such that

\[
\begin{pmatrix}
\dot{x}_a(t_k^+) \\
\dot{y}_a(t_k^+)
\end{pmatrix} = \begin{pmatrix} \sigma \\ 0 \end{pmatrix}
\]

(5.100)

\( \sigma > 0 \Rightarrow P \in D_+ \)

(5.101)

\( \sigma < 0 \Rightarrow P \in D_- \)

\( \sigma = 0 \Rightarrow P \in \mathcal{C} \)

where \( \mathcal{C} \) denotes the friction cone (real-world), \( D_+ \) and \( D_- \) are the two half-lines delimiting it.

These relationships are the classical Coulomb’s law of friction with the additional assumption in (5.100) \(^{23}\). Now (5.99) can be rewritten as (taking (5.100) into account)

\[
P = -H^{-1} \begin{pmatrix}
\dot{x}_a(t_k^-) \\
\dot{y}_a(t_k^-)
\end{pmatrix} + \sigma H^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

(5.102)

Equation (5.102) simply expresses that the percussion vector \( P \in \mathbb{R}^2 \) belongs to the line \( \Delta \) with directing vector \( H^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), passing through the point \(-H^{-1} \begin{pmatrix}
\dot{x}_a(t_k^-) \\
\dot{y}_a(t_k^-)
\end{pmatrix}\). \( \Delta \) intersects the axis \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) at the origin when \( \dot{y}_a(t_k^-) = 0 \), and is orthogonal to the vector \( H^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{\mu}{\mu} \sin(\theta) \cos(\theta) \\ \mu^2 \cos^2(\theta) \end{pmatrix} \).

The possible outcomes of the process depend on the relative position of \( \Delta \) with respect to \( \mathcal{C} \). It is shown in [381] that the stated evolution problem always possesses at least one solution, possibly nonsmooth. In particular, if \( \Delta \supset D_+ \) or \( \Delta \supset D_- \), which means that \( P \in \partial \mathcal{C} \), then (5.100) (5.101) imply that

\[
\begin{pmatrix}
\dot{x}_a(t_k^+) \\
\dot{y}_a(t_k^+)
\end{pmatrix} = \begin{pmatrix} \sigma \\ 0 \end{pmatrix} \in \mathcal{C}
\]

\(^{23}\)Note that similarly as for Baraff’s rules, the constraint that \( \dot{y}_a(t_k^+) = 0 \) is consistent with the intuitive fact that the normal force cannot provide any work.
[0_2, A], where 0_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} and A = \begin{pmatrix} \dot{x}_a(t_k^-) \\ \dot{y}_a(t_k^-) \end{pmatrix} = \begin{pmatrix} \dot{x}_a(t_k^-) \\ 0 \end{pmatrix}, i.e. there may be a jump in \( \dot{x}_a \) \(^{(24)}\). Note that \( \dot{\theta}(t_k^+) \) can be computed from (5.92) and \( \sigma, \dot{x}_a(t_k^-) \) \( \dot{y}_a(t_k^-) = 0 \). It can be easily verified that a jump may occur when \( \mu = \mu_c \), where \( \mu_c \) is defined in (5.98). In this latter case, the slope of \( \Delta \) and that of \( D_- \) are the same, i.e. \( \frac{1}{\mu_c} \).

Another situation when a discontinuous velocity can be a solution of the dynamics is when \( \Delta \) intersects the interior of \( C \). Then one has to check whether it is possible for \( P \) (see (5.102)) to belong to \( \text{Int}(C) \) also. Looking at conditions (5.101) it follows that this implies that \( \sigma = 0 \), i.e. \( \begin{pmatrix} \dot{x}_a(t_k^-) \\ \dot{y}_a(t_k^-) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \) i.e. sliding stops and sticking occurs. Thus \( P \in \text{Int}(C) \) implies \( -H^{-1} \begin{pmatrix} \dot{x}_a(t_k^-) \\ \dot{y}_a(t_k^-) \end{pmatrix} \) \( \in \text{Int}(C) \). In other words, one has to check whether the vector \( -H^{-1} \begin{pmatrix} \dot{x}_a(t_k^-) \\ \dot{y}_a(t_k^-) \end{pmatrix} \) lies inside \( C \): if it is the case then there may be a velocity discontinuity and a tangential shock. But \( P = 0 \) with a continuous velocity is also a solution in this case. Most importantly, notice that it is not a priori assumed that the system enters a sticking regime when there is an inconsistency. In other words, the method presented in [381] does not preclude continuation of the sliding regime. Finally when \( \Delta \cap C \) reduces to the origin of the local frame, then the unique solution is a continuous velocity, i.e. \( \sigma = \dot{x}_a(t_k^-) \) (trivially then from (5.102) \( P = 0 \)). The different situations are depicted in figure 5.14.

In the slender rod example, there are therefore three possibilities: since \( \Delta \) intersects the origin, either its slope makes it lie entirely inside \( C \), or it lies entirely on the boundary of \( C \) (\( D_+ \) or \( D_- \)), or it lies entirely outside \( C \) (except for the origin).

In summary, whatever the preimpact conditions may be, there is always a \( \sigma \) and a percussion vector \( P \) that satisfy (5.100) (5.101). Depending on the sign of the found \( \sigma \), there may be sticking, sliding with slip reversal or not. Conditions i, ii, iii listed at the end of the foregoing subsection are satisfied. The algorithm also allows one to treat the case of soft collisions, see example 4.1 in chapter 4.

It is noteworthy in (5.100) (5.101) (5.102) that the percussion vector is deduced from the sign of the postimpact velocity. This is an arbitrary choice coming from the assumption that the postimpact velocity remains tangent to the constraint. It is also remarkable that this formulation lends itself very well to numerical investigations [245] [380]. Some numerical results clearly display the apparently non-deterministic nature of the discontinuities in \( \dot{x}_a \), see [381] §15. These numerical result are consistent with the fact that \( \dot{x}_a \) grows unbounded before a catastrophe, which implies that \( F_n \) does, hence \( B(\theta, \mu) \to 0 \), or equivalently \( \mu \to \mu_c \). Also it is apparent that after the IW/OC, the system reenters a consistent regime, i.e. condition iv is satisfied.

\(^{24}\) The case \( \sigma = 0 \) is called a maximal catastrophe in [381].
Results are obtained for the Painlevé’s example, with additional external actions on the rod.

Conclusions

In conclusions to this subsection on tangential impacts, the problem of finding an IW/OC rule that enables one to calculate velocity jumps when the system has attained an inconsistency of the associated LCP, and at the same time allows to guarantee condition iv, remains open (25). We also conjecture that the Painlevé’s example with no external action except gravity, is always consistent. In other words, if the system we described in this subsection is initialized in a sliding regime, then it attains no inconsistent configuration until it sticks at rest on the horizontal rigid surface. Finally, as noted in [344], Because various distinct phases (slip, no contact ...) may be encountered along such fast transitions and because friction effects involve path-dependent dissipation, the determination of the final state of an instantaneous jump will in general require that the evolution of the system along the instantaneous jump must also be followed. Although this sentence was formulated in the framework of displacement jumps in certain quasistatic problems with friction, it could well be that it applies also to IW/OC.

5.4.4 Additional comments and bibliography

The classical example of the chalk [381] [125] [245] illustrates nicely the above developments. This is a very simple experiment anyone can do: take a piece of chalk and make it slide on a blackboard, while applying a normal and a tangential forces on it. Then for a suitable velocity, orientation and applied force, the chalk begins to alternatively slide, stick, and jump on the board’s surface. The sound indicates clearly

\[ u'' = \left( \frac{dx}{dt}, \frac{dy}{dt} \right) \]

\[ \Delta (\mu > \mu_c) \]

\[ \text{arctan}(1/\mu_c) \]

Figure 5.14: Moreau’s algorithm (planar system with friction).

\[ \text{Painlevé’s example} \]

\[ \text{Results are obtained for the Painlevé’s example, with additional external actions on the rod.} \]

\[ \text{Conclusions} \]

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\[ \text{It would however be interesting to discover a case such that the system attains an inconsistent configuration, and with } \mu < \mu_c \text{ so that the percussion vector needed for a velocity jump lies outside the friction cone, i.e. to prove that such a case does exist.} \]
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that there is a collision each time contact is re-made, and the chalk leaves on the board a dotted line. The possible jump in \( \dot{x}_a \) and \( \dot{\theta} \) create new initial conditions that may in effect induce a subsequent detachment of the rod from the surface \( y_a = 0 \). The chalk example may be modeled with a rod, acted upon by some external forces and torques.

Consideration of static friction can be found in [32] §9, using a graphical method from [572]. Wang and Mason [572] show that in certain cases, the consideration of an impulsive force is the only possible solution to this dynamical problem, which otherwise possesses no solution at all, thus retrieving a result by Pères in [429] chapter 10 based on a graphical analysis of a 2-dimensional shock process. The 3-dimensional case with friction is also treated in [429], using Delassus’ lemma. Other graphical analysis can be found in [135].

Some authors [136] emphasize the case \( B(\theta, \mu) = 0 \), when \( \mu = \mu_c \) for the critical value defined in (5.98). Indeed, this value of the friction coefficient is such that some extended inertia matrix, obtained by incorporating \( F_n \) in the generalized acceleration vector (and assuming that \( F_t = \mu F_n \)), is singular. It is claimed in [136] that for this particular value of the friction coefficient, there must be velocity discontinuities, based on the observation that \( F_n \) in (5.90) then tends to infinity. The preceding developments prove that this statement is not complete. The \( \text{IW}/\text{OC} \) percussion vector does not necessarily lie on the boundary of the friction cone, but may lie inside of it. From the calculation of \( F_n \) for the dynamics in sliding regime in (5.90), one sees that if \( \mu \to \mu_c \), \( F_n \) does not necessarily diverge since the numerator may also tend to zero. The same authors choose to study an approximating problem in [137], using singular perturbations techniques to investigate what happens when the stiffness and damping tend to infinity (in a way similar to what we have developed in chapter 2). Basically the goal is to discover whether there is a relationship between the problem of existence of solution of the LCP, and the stability of the so-called boundary layer system obtained in the singular perturbation analysis. Once again the approximating problems method (or \textit{genetic} method as it is called in [292]) seems to be a practical and elegant way to study rigid body dynamics.

Sinitsyn [496] studies this problem from the definition of 3 surfaces: let \( T \) be the body’s twist, \( W = [F^T, C^T] \) the impulsive wrench of external actions. Then if \( q \) is a set of generalized coordinates, we get \( T = Mq, \sigma_q = M^{-1}M^T \sigma_T, \) thus \( \sigma_T = MM^{-1}M^T W = \tilde{M}^{-1} \sigma_T \), see section 4.1. From the expression \( \sigma_q^T M \sigma_q = W^T \tilde{M}^{-1} W = 2\Phi_1 + 2\Phi_2 + 2\Phi_3 \), [496] extracts the 3 functions \( \Phi_1 = \frac{1}{2}F^T AF, \Phi_2 = \frac{1}{2}CTBC, \Phi_3 = \frac{1}{2}F^TC^CC \) and associates 3 surfaces to each of them (two are ellipsoids, the third may be a hyperboloid). Then [496] derives conditions on the matrices \( A, B \) and \( C \) such that collision without impact can occur, the main conclusion being that rolling friction is the primary cause of this phenomenon. The analysis developed in

---

\(^{26}\)Let \((S)\) be a \(n\)-dimensional multibody system with generalized coordinates vector \( q \), in contact through its part \( S_1 \) with a body \( S_2 \) at the point \( I \). Let \( X = J(q)\dot{q} \) be the velocity of \( I \) in a frame centered at \( I \) (A task-space frame in robotics language). Then Delassus’s lemma states that \( X = \nabla_{u,v,w} \Phi_q(u,v,w) \) for some quadratic function \( \Phi \) of the interaction force components \( u, v \) and \( w \) in the same frame.
[496] does not straightforwardly apply to the planar case we develop above since it relies on the definition of the three surfaces $\Phi_i$. How can they be defined when the wrench reduces to two force components and one moment component?

Matrosov and Finogenko [350] analyze $n$-degree-of-freedom systems subject to unilateral constraints of the form $q_i \geq 0$ and assume that Coulomb's friction law applies directly to these constraints (like in the $z$-coordinates in (5.78)). Then the authors derive a condition which asserts that if the friction coefficient is small enough, then the mapping which allows to calculate $\dot{q}$ from the dynamics is contractive, hence there is a unique generalized acceleration. From this point the dynamics admit a unique classical solution. The rod example is treated and the condition $\mu < \mu_c$ is found to assure existence and uniqueness of a solution. These results are hence of the same nature as those in [543] who establish existence and uniqueness of a solution to NCP's when the friction coefficient is small enough. Jean and Pratt [246] also derived sufficient conditions guaranteeing the local existence of a solution with continuous position and velocity. They show on an example that the considered evolution problem possesses in general a solution only if discontinuous velocities are admitted, because there is a set of parameters and initial data such that the normal interaction force is infinite. Pozharitskii [443] studies the same problem and extends Gauss' principle to systems with unilateral constraints and dry friction. The inconsistency problem is related to the phenomenon called jamb in [519], see [319]. Jamb occurs in impacts where the friction coefficient (more exactly the impulse ratio) $\mu$ is larger than some critical value $\bar{\mu}$. $\bar{\mu}$ is such that the normal acceleration has the right sign when one computes it assuming that after sticking, the direction of the subsequent slipping motion is directly opposite to that of the tangential impulse. This is based on the assumption that the collision duration is strictly positive, see subsection 4.2.4. Strange behaviour of the models with friction can also be given an explanation via the transportation of real-world friction cones into the configuration space [139] [380], see remark 6.25. When the configuration-space friction cone (i.e. the image of the 3- or 2-dimensional real-world cone in the configuration space) dips below the tangent space $TV_\varphi$, the model may possess zero, one or more solutions.

Most importantly, let us recall that Monteiro-Marques [366] chapter 3, proves the global existence of a solution to the sweeping process problem with dry friction with velocity $RCLBV$. This shows that within the space of motions whose velocities are $RCLBV$, there is at least one element that satisfies the dynamical equations of the system. Note that this however does not help us to answer to the question: is a discontinuity in the velocity going to occur or not when a sliding regime is initialized?

### 5.4.5 Numerical investigation of impacting systems

Let us end this chapter with few words on the problem of numerical integration of mechanical systems subject to unilateral constraints and impacts. The main difficulty is that the dynamics of such systems are not continuous, but rather of
intermittent type. Algebraic constraints and shock conditions have to be taken into account. Notice at once that if the system is subject to an infinity of impacts and if the impact Poincaré map is explicitly calculable, then one obtains a discrete-time system, which is easily simulated by iterations. Unfortunately, even for very simple systems, it may be impossible to obtain the Poincaré map, see subsection 8.5.2. Some authors [393] [295] [392] develop analytical methods based on the series expansion of the quasi-coordinate, which allow to obtain a system of nonlinear finite-difference equations relating the velocities and the flight-times at the collision times. An approximating value of the impact sequence duration is given in [393]. Such results could be used to calculate numerically the motion of certain impacting systems, since they yield an approximation of the dynamics as represented in (7.42).

The most basic idea is to integrate the motion between impacts, i.e. on intervals \((t_k, t_{k+1})\), to determine the impact times, compute the velocity jump, and start again the integration of the free-motion vectorfield. This is known as the method of fitting. The main drawback of this naive method is the length of simulation, that may become very large if for instance several constraints are attained. But for simple systems, it may prove very useful. Any standard numerical integration can be used between impacts. The collisions detection can be done by decreasing the integration step in the neighborhood of the constraint surface. It has been investigated for instance in [190] [579] [591] [592] [152], in which general numerical algorithms for systems of rigid bodies with unilateral constraints are presented, based on DAE (differential-algebraic equations) methods for constrained systems. Other methods based on the use of the quasi-coordinates \(^{27}\) have been presented in [573] [89] [371] [170]. In particular it is emphasized in [573] that if an explicit form of the impact Poincaré map is available (see chapter 7), then one can advantageously use this discrete-time system to compute the motion during repeated collisions, hence avoiding the drawback of having to integrate with smaller and smaller steps, not to miss collisions.

The second "natural" idea is to replace the constraint by a compliant model (see chapter 2). Then one has only to integrate ODE's. The main drawback, as we already noticed, is that this yields in general stiff ODE's. Therefore the integration step has to be reduced and the simulation length becomes large. On the other hand, when dealing with complex systems, it may not be evident to determine which effects the flexibilities have on the motion of the system, and how they have to be chosen to obtain results close to the real motion, see e.g. [437].

Recently, Paoli and Schatzman proposed a specific numerical scheme devoted to the integration of systems with unilateral constraints [417] [415]. The nice feature of the proposed discretization is that it is proved that the solution of the approximating problem converges uniformly towards that of the original system in proportion as the step converges to zero. This is a very strong property, as it allows one to be sure that

\(^{27}\)which basically are found by posing \(q_0 = f(q)\) if \(f(q) = 0\) is the constraint surface. They therefore represent directly the "distance" between the bodies, and are widely used in the mechanical literature.
the obtained motion is close to that of the system, and that the observed dynamical phenomenons really pertain to the system, not to the discretized approximation.

Let us mention, in relationship with the sweeping process formulation presented in chapter 5, that powerful numerical algorithms based on the discretization of the sweeping process, are presented in [380]. Interestingly enough, those discretization tools have been used in [366] to prove existential results. They have also proved to provide good prediction of robot motion in [437] [546], where experimental results are presented.

Let us end this short review by noting that softwares dedicated to the study of dynamical systems (that allow to investigate numerically the existence of chaotic motion, periodic trajectories ...) are restricted to either continuous-time or to discrete-time (mappings) systems. It is clear that one cannot use such softwares directly for systems with unilateral constraints, unless the impact Poincaré map is available explicitly. The Zhuravlev-Ivanov change of coordinates (see chapter 1) may then prove to be very useful, since it allows one to obtain a time-continuous system. This path has received some attention in [349]. The major drawback is the limitation of the Zhuravlev-Ivanov method to codimension one constraints. This is not the case for the other cited methods that can handle that case.
Chapter 6

Generalized impacts

In this chapter, we analyze the rigid body impact problem from a generalized point of view. The main ideas are the following:

i) Since a Lagrangian system is a point moving in its configuration space, it is natural to apply contact-impact rules using this point of view. More clearly, unilateral constraints define domains of the configuration space, and the point that represents the system strikes the boundaries of these domains (we already took this point of view in chapter 5, section 5.3). At the times of these "generalized" collisions, one needs additional informations to be able to calculate the postimpact values (recall that this is a basic fact of nonsmooth impact mechanics, see e.g. (4.25)): these may be named generalized restitution rules. Notice that this has to be considered exactly at the same level as the sweeping process formulation described in chapter 5: one provides a particular evolution problem, and tries to prove its theoretical coherence and practical usefulness if possible. In the same way as the sweeping process formulation extends purely inelastic impacts to complex systems, generalized restitution rules in the configuration space extend restitution rules for particles to more complex Lagrangian systems.

Moreover, this chapter is devoted to clarify and explain the use of the kinetic metric in the definition of restitution laws for rigid body impacts. Let us recall that the relationship derived in example 1.2:

\[ M(q)\sigma_q(t_k) = p_q \]  

is always valid (\( p_q \) denotes the generalized percussion for coordinates \( q \)). But still unilateral constraints problems imply more than this, namely restitution rules. As we saw, the problem is not trivial even for two rigid bodies striking. We shall discuss the possible extension of Newton's rule to more complex systems. This study is important in view of control of mechanical systems with unilateral constraints (manipulators, bipedal robots, juggling robots, mechanisms with clearances...), that will be treated in chapter 8.

ii) The second message we would like to transmit is that it may be dangerous (i.e. yield wrong conclusions) to apply some rules a priori supposed to be general (or
universal). More precisely, consider the example of a sphere colliding a frictionless obstacle: then it is known from Newton’s rule applied to the normal contact velocity and from the dynamics that the tangential velocity remains continuous at impact while the normal one suffers from a discontinuity. Then one could be tempted to extend this to more complex systems. As we shall see, this is a totally wrong way of proceeding. A second widely spread rule is that of conservation of momentum that holds in certain situations: the momentum after the shock is the same than that before the shock. Should one apply such a rule? Is it really a rule? In fact, we shall see that the only rules one can apply are restitution rules along a certain direction (a generalized normal direction), and then calculate the rest of the postimpact variables with the shock dynamical equations. Nothing else seems reasonable. Any other rule is just a consequence of that. Note again that Brach’s philosophy, that we described in subsection 4.2.2, fits exactly with this one: one must write down the percussion dynamical equations. Then one tries to render the system solvable (i.e. postimpact velocities and impulse magnitude can be computed, there are as many unknowns as equations) by adding some restitution rules.

6.1 The frictionless case

6.1.1 About "complete" Newton’s rules

It is sometimes accepted that the velocities that suffer from discontinuities are those which are orthogonal to the constraint surface \( f(X) = 0 \), where we assume for the moment that \( f(\cdot) \) is scalar, whereas the tangential part remains continuous. In the following we shall name a complete Newton’s restitution rule such a contact law. As we shall see, this reasoning poses two problems: first it is to be precisely defined what is meant by orthogonality, and with respect to which surface it is to be understood. Second, the continuity of the tangential velocity should not be a priori set, but is rather a consequence of the shock dynamics. Assume that in a set of generalized coordinates \( X \), we have \( \dot{X}_n(t_n^+) = -e\dot{X}_n(t_n^-) \), where \( \dot{X}_n \) denotes the component of \( \dot{X} \) along the constraint normal \( \nabla_x f(X) \), whereas the remaining components are denoted as \( \dot{X}_t \). This may correspond to the task-space coordinates defined in [606], and posing \( X_n = f(X) \) the constraint is \( X_n \geq 0 \) (Recall that such \( X_n \) is called a quasi-coordinate). If we apply Newton’s rule directly to the normal components of \( \dot{X} \) while considering that the tangential ones remain continuous, we obtain a relationship like \( \dot{X}(t_k^+) = \mathcal{E} \dot{X}(t_k^-) \) for some diagonal constant matrix \( \mathcal{E} \) with entries \( e \) and 1. Now let \( \dot{q} \) be another vector of generalized coordinates, with \( \dot{X} = J(\dot{q})\dot{q}, \ J \) the full-rank Jacobian. Then \( \nabla_q f o X(q) = J^T \nabla_x f(X) \), and if \( \dot{X}^T \nabla_x f(X) = 0 \) then \( \dot{q}^T \nabla_q f o X(q) = 0 \). However we get also \( \dot{q}(t_k^+) = J^{-1} \mathcal{E} J \dot{q}(t_k^-) \), and clearly in general \( J^{-1} \mathcal{E} J \neq \mathcal{E} \), and there is no reason why the discontinuous part of \( \dot{q} \) should be colinear to \( \nabla_q f o X(q) \). Consequently a complete Newton’s rule is not in general coordinate invariant if applied in any "task-space" frame. In the following we shall see that the complete Newton’s rule is in fact true in a particular task-space frame. Moreover
6.2  THE USE OF THE KINETIC METRIC

Figure 6.1: Two masses colliding.

the use of these particular velocity coordinates will make it clear to which velocity coordinates a restitution rule must be applied, and which ones remain continuous.

As the following example shows, neglecting inertia in the definition of the normal direction can also yield other wrong conclusions if one applies directly the complete restitution rule by a priori assuming continuity of some tangential components, besides non-invariance by coordinates change. Thus the second (false) assertion of such a rule yields in general wrong conclusions, because it imposes artificially an impulsion at the contact point that actually does not exist. This is easily seen in (4.25): if the percussion vector has a certain form, we can see from (4.44) that we have to define as many restitution coefficients as components in P. On the other hand, if one sets that for instance \( \sigma_{V_{\alpha}}(t_k) \) is of a certain form, this implies that the percussion is modified as well.

Example 6.1 Let us consider two rigid bodies moving on the same line, with respective coordinates \( x_1 \) and \( x_2 \) as depicted in figure 6.1. Assume the relative distance between them is \( x_0 = x_2 - x_1 \). Thus the unilateral constraint is \( x_0 \geq 0 \). Let us denote \( x^T = (x_1, x_2) \) and \( \dot{x}^T = (x_1, x_0) \). It is noteworthy that even in this quite simple case, applying Newton's rule to the velocities along the Euclidean normal to the constraint yields erroneous conclusions. Indeed suppose that \( \dot{x}_n = \dot{x}^T N_2 N_x \) with \( N_x = \frac{\nabla f(x)}{\|\nabla f(x)\|} \). Then \( \dot{x}_n^+ = -e \dot{x}_n^- \) yields \( \dot{x}_0^+ = -e \dot{x}_0^- \), but applying the rule \( \dot{x}_t^+ = \dot{x}_t^- \) yields \( \sigma_{x_1} + \sigma_{x_2} = 0 \), which is not true except if \( m_1 = m_2 \). Thus the wrong conclusions stem from a priori asserting continuity of \( \dot{x}_t \) at the percussion time, a property that should instead come from the dynamical equations. We shall re-analyze this example in the following when we have explained the use of the kinetic metric for restitution rules.

6.2  The use of the kinetic metric

As stated in [232] [233] [292] (see also [378] in the sweeping process formulation, see also [32] and references therein) one can understand orthogonality in the tangent space to the configuration space in the sense of the metric of the kinetic energy (thus generalized coordinates \( q \)-invariant), i.e. in the sense of the scalar product \( x^T M(q) y \). This simply means that one no longer works like if velocities were elements of the Euclidean space \( \mathbb{R}^n \), but considers that they belong to the tangent space to the configuration space of the system, equipped with the kinetic energy metric.

The kinetic metric choice is quite natural if one considers the shock dynamic
equation \( M(q)\sigma_q = p_q \), that yields \( \sigma_q = M^{-1}(q)p_q \). Hence it is clear that the discontinuous part of the velocity is along the vector \( M^{-1}(q)p_q \), which in turn is colinear to \( M^{-1}(q)\nabla_q f(q) \) for frictionless constraints (Indeed the interaction force is along the Euclidean normal from the virtual work principle). We thus define the normal direction to a constraint \( f(q) = 0 \) as being given by \( \frac{M^{-1}(q)p_q}{\sqrt{\nabla_q f(q)^T M^{-1}(q)p_q}} \) \( \text{def. 2.5.14} \), while the tangential one is orthogonal (in the Euclidean metric sense) to \( \nabla_q f(q) \). Thus the postimpact velocity will generally depend on the system configuration at impact. Now it can be verified that if \( \dot{q}_n(t_k^+) = -e\dot{\bar{q}}_n(t_k^-) \) with \( \dot{q}_n \triangleq \dot{q}^T M(q)n_qn_q \), where the unitary normal vector is \(^1\) then \( \dot{X}_n(t_k^+) = -e\dot{X}_n(t_k^-) \) since \( J \) is square full-rank. One deduces also that

\[
(\dot{q}(t_k^-))^T \nabla_q f(q) = -e(\dot{q}(t_k^-))^T \nabla_q f(q)
\]  

(6.3)

Thus one concludes that the components of \( \dot{q} \) to which the restitution rule has to be applied are the same as if the Euclidean metric had been used. We shall however rapidly see the usefulness of the kinetic metric. The generalized contact percussion vector is given by (see example 1.3 in chapter 1, equation (1.12)):

\[
p_q = J^T p = -(1 + e)\frac{(\dot{q})^T \nabla_q f}{\nabla_q f^T M^{-1}(q)\nabla_q f}
\]

(6.4)

and is thus along \( \nabla_q f \). A vector of generalized velocities \( \dot{q} \in \mathbb{R}^n \) may thus be expressed in a frame \( (n_q, t_q) \), where the \( n-1 \) unitary vectors \( t_{qi} \) are chosen mutually independent and such that \( t_{qi}^T M_q n_q = 0 \), i.e. \( t_{qi}^T \nabla_q f(q) = 0 \).\(^2\)

Let us now consider the dynamics of a rigid manipulator as in (1.10). We define the full-rank \( n \times n \) matrices \( \Xi = \begin{bmatrix} n_q^T \\ t_q \end{bmatrix} \) and \( \mathcal{M} = \Xi M(q) \), and the vector of velocities

\[
\mathcal{M}\dot{q} = \begin{bmatrix} \dot{q}_{\text{norm}} \\ \dot{q}_{\text{tang}} \end{bmatrix}
\]  

(6.5)

Then the dynamics can be written as

\[
\begin{cases}
\dot{q}_{\text{norm}} - E_1 \mathcal{M}\dot{q} + n_q^T (C(q, \dot{q}) \dot{q} + g(q)) = n_q^T (u + p_q \delta_t) \\
\dot{q}_{\text{tang}} - E_2 \mathcal{M}\dot{q} + t_q^T (C(q, \dot{q}) \dot{q} + g(q)) = t_q^T u
\end{cases}
\]  

(6.6)

where \( E_1 = [I_m; 0_{m \times n-m}] \), \( E_2 = [0_{n-m \times m}; I_{n-m}] \), \( p_q \) is the generalized percussion vector, and we use the fact that \( t_q^T p_q = 0 \). These velocities coordinates possess

\(^1\)To simplify the notations, we write \( \nabla_q f(q) \) instead of \( \nabla_q f q X(q) \) in the following.

\(^2\)Similarly to [606], we define a basis is the tangent space \( T_q \) to the configuration space at \( q \), but we endow \( T_q \) with the kinetic metric. This is different in general from [324] who propose a generalized coordinates transformation.
6.2. THE USE OF THE KINETIC METRIC

the advantage compared to other sets of velocities that it clearly appears which components are discontinuous and which ones are continuous. \( \dot{q}_{\text{norm}} \) and \( \dot{q}_{\text{tang}} \) are the components of \( \dot{q} \) along \( n_q \) and \( t_q \) respectively (Recall that due to the "generalized" scalar product we have chosen, the projection of a vector \( x \) on a unitary -in the corresponding norm- vector \( y \) is given by \( x^T M(q) y \), so that in an orthonormal basis \( (y, z) \) we have \( x = x^T M(q) y + x^T M(q) z = x_y y + x_z z \).

From (6.6) and Newton’s rule applied to \( \dot{q}_{\text{norm}} \) it follows that at the impact times \( \sigma_{\text{norm}}(t_k) = -(e + 1) \dot{q}_{\text{norm}}(t_k^+) = n_q^T p_q = -(e + 1) \dot{q}_{\text{norm}}(t_k^-) \) whereas \( \sigma_{\text{tang}}(t_k) = 0 \), thus these new coordinates allow one to separate discontinuous components \( \dot{q}_{\text{norm}} \) from continuous ones \( \dot{q}_{\text{tang}} \) \(^3\). It thus clearly appears why it is necessary to apply a restitution rule to the components along \( \nabla_q f(q) \), and only to those ones. As said above, continuity of the tangential part is a consequence of the dynamics, see (6.6). Finally note that this manipulation can be performed on any Lagrangian system, and is not restricted to rigid manipulators.

**Remark 6.1 On manipulator feedback control**

If the controller \( u \) is suitably designed so that the impact sequence \( \{t_k\} \) is infinite, then \( \dot{q}_{\text{norm}} \) vanishes in finite time for \( 0 \leq e < 1 \) (This controller should obviously be only one part of a more complex control strategy involving switching times between several basic controllers for free-motion, impact and constrained-motion phases, see chapter 8). This can easily be done via a linearizing state feedback plus an additional constant negative force control, as \( u = C(q, \dot{q}) \dot{q} + g(q) - \Xi^{-1} M(q) t_q \ddot{v} \) which results in the closed-loop system

\[
\begin{align*}
\{ \dot{q}_{\text{norm}} \} &= n_q^T v \\
\dot{q}_{\text{tang}} &= t_q^T v
\end{align*}
\]

The acceleration is put between brackets to emphasize that the controller can influence its evolution only outside the impacts, i.e. for \( t \neq t_k \) (see chapter 1). In order to guarantee that the impact sequence does exist, it is sufficient to insure a negative relative normal acceleration between the two bodies \[^{[574]}\], i.e. \( \dot{q}_{\text{norm}} < 0 \). Let us choose \( v = -\nabla_q f(q)^T \sqrt{\nabla_q f(q)^T M(q) \nabla_q f(q) + M(q) t_q} \ddot{v} \), so that (6.7) becomes

\[
\begin{align*}
\{ \dot{q}_{\text{norm}} \} &= -1 \\
\dot{q}_{\text{tang}} &= \ddot{v}
\end{align*}
\]

where we assume that the vectors \( t_q \) have been chosen mutually orthogonal in the kinetic metric sense. Equation (6.8) together with Newton’s restitution law

\[^{3}\text{Consider that the constraint is given by } q_1 = f(q) \geq 0. \text{ Then } n_q = e_1 \text{ and } t_q,i = e_i, \text{ for } i = 2, \ldots, n, \text{ where } e_i \text{ is the } i\text{th unit vector. It follows that the generalized momenta } p_i = \frac{\partial}{\partial q_i} = t_q^T M(q) \dot{q} \text{ are continuous. The } p_i's \text{ are nothing else than what we have denoted } \dot{q}_{\text{tang},i}. \text{ The fact that these quantities remain continuous at collision times was first noticed by Appel [14]. See also [526] who retrieves the fact that } \dot{q}_{\text{tang}} \text{ is time-continuous ([C] in [526] is exactly } t_q \text{ and equation (2.3)" in [526] states that } \sigma_{\dot{q}_{\text{tang}}} = 0).
implies that $\dot{q}_{\text{norm}} \to 0$ in finite time for $\varepsilon \in [0, 1)$. Since the impacts occur for $f(q(t_k)) = 0$, and since the applied normal force is kept strictly negative, the robot's tip is stabilized in the normal direction at $f(q) = 0$. From (6.8) it is easy to design $\tilde{v}$ such that the tangential velocity $\dot{q}_{\text{tang}} \to 0$ asymptotically. But what about the tangential displacements? If we assume that $\dot{q}_{\text{tang}}$ is integrable, then we can solve the position control problem. But if it is not integrable, one has either to leave the robot's tip evolve until $t_{\infty} (\triangleq +\infty)$, or to try another set of vectors $t_{q_i}$'s. Mimicking [606] we could assume the existence of $s \in \mathbb{R}^n$ such that $\dot{s} = M\dot{q}$, hence the whole robotic task control might be synthesized in $s$-coordinates. It however remains to be shown that such $s$ actually exists, i.e. find out conditions such that $M$ is a Jacobian. It may then be better to use other vectors $n_q$ and $t_q$ that may not be unitary in the kinetic norm, but simply independent and if possible orthogonal one to each others (this is what is proposed in [607] to define the task-space coordinates).

Remark 6.2 Flexible joint manipulators, continued

In section 3.3.1, we investigated the shock dynamical equations for two models of flexible joint manipulators. Since the dynamical equations of such systems can be written similarly as for rigid manipulators (only the number of degrees of freedom is changed, as well as the torque input vector), it is clear that the coordinate change in (6.6) can also be performed for such systems. The difference between the two models in (3.11) and (3.13) is that the inertia matrix contains cross-terms in the second case. Notice that the vectors $t_q$ do not depend on the inertia matrix, but only on the constraint manifold normal direction. In the same way, although $\dot{q}_{\text{norm}}$ depends on the inertia (see the denominator of $n_q$), this does not influence the restitution rule. Indeed the denominator is only position-dependent, hence continuous. The restitution rule is therefore in both cases $\nabla_J(q_1)\dot{q}(t_k^+) = \frac{\partial f}{\partial q}^T \dot{q}_1(t_k^+) = -e^T \dot{q}_1(t_k^+)$. However the definition of $\dot{q}_{\text{tang}}$ is modified since $\dot{q}_{\text{tang}} = t_q^TM(q)\dot{q}$ and $t_q$ is a priori independent of $M(q)$. We conclude that the velocity components that remain continuous at the impact times are not the same in the simplified model [511] and in the complete model [539]. Note that it might happen, for certain constraints and restitution coefficients, that the motorshaft velocities remain continuous, but this does not seem to be the case in general.

Remark 6.3 Kecs and Teodorescu [273] §6.1.2 develop a theoretical framework on purely inelastic collisions. They consider the equality between momenta $m_i\dot{q}_i$ of particles as an equivalence relation, and define internal and external composition laws. It turns out that these laws correspond to inelastic shocks. Then they construct the corresponding quotient space of plastic collisions, and endow it with a distance, which is precisely the kinetic energy of an equivalent class (i.e., the kinetic energy of any of the elements in the class, which all possess the same kinetic energy by definition of the equivalence relation). Note that although the construction of such a metric space is is an elegant mathematical framework, its usefulness remains to be proved.
6.2.1 The kinetic energy loss at impact

Let us compute the kinetic energy in the basis \((n^T_q, t^T_q)\):

\[
T = \frac{1}{2} q^T M_q \dot{q} = \frac{1}{2} \dot{q}^T \mathcal{M}^{-1} \dot{q} = \frac{1}{2} \dot{q}^T \mathcal{M}^{-1} \dot{q}
\]

Using the definitions of \(n_q\) and the \(t_{q,i}\)'s, it follows that

\[
\Xi M_q \Xi^T = \begin{bmatrix}
1 & 0 \\
0 & t^T_q M_q t_q
\end{bmatrix}
\]

Hence

\[
T = \frac{1}{2} \dot{q}^2_{\text{norm}} + \frac{1}{2} \dot{q}^T \Xi \Xi^T \dot{q}
\]

If the \(t_{q,i}\)'s are chosen mutually orthogonal (in the kinetic metric sense) then the term \(t^T_q M_q t_q\) is the \((n-1)\times(n-1)\) identity matrix, and we get the simple expression for the kinetic energy:

\[
T = \frac{1}{2} \dot{q}^2_{\text{norm}} + \frac{1}{2} \dot{q}^T \dot{q}_{\text{tang}}
\]

Thus it can be checked that in this case

\[
T_L(t_k) = \frac{1}{2} (e^2 - 1) \left( q_{\text{norm}}(t_k) \right)^2 \leq 0
\]

an expression which generalizes \(T_L\) for two bodies colliding, see (4.41). Recall that the constraint \(f(q) \geq 0\) can represent a sort of distance between the bodies, so that \(q_{\text{norm}}\) may represent the relative normal velocity, that reduces to \(v_{2,n}\) in (4.41). Therefore the assertion that \(0 \leq e \leq 1\) is true for frictionless percussions in general. A wrong definition of the normal direction could have led us to conclude that \(e \geq 1\) was possible: indeed consider the \(\dot{q}\)-space. Then \(T_L \leq 0\) implies that \(T = T(t_k)\) defines a level set that contains the level set of \(T(t^*_k)\). But a priori nothing would prevent that \(|\dot{q}(t_k^*)| > |\dot{q}(t_k^*)|\), since the negative jump in \(T\) may be due to a change of the direction of \(\dot{q}\).

Remark 6.4 The use of the transformed velocity and the simple form of the kinetic energy loss at impacts allows us to retrieve a result by Kirgetov [280] which states that after an elastic collision, the state of the system satisfies \(\nabla_q f(q) \dot{q}(t_k^*) = -\nabla_q f(q)^T \dot{q}(t_k^*)\), and that is differs from other possible states consistent with this condition in that it also satisfies \(\sigma(t_k) M(q(t_k)) \delta q = 0\), where \(\delta q\) is a virtual displacement subject to \(\nabla_q f(q)^T \delta q = 0\).

From (6.13) one deduces that if an impact occurs (i.e. if \(q_{\text{norm}}(t_k^*) \neq 0\), then \(T_L = 0\) if and only if \(e = 1\), i.e. \(q_{\text{norm}}(t_k^*) = -q_{\text{norm}}(t_k^*)\). Moreover, \(\sigma(t_k)\) is orthogonal to \(f(q) = 0\), i.e. \(\sigma(t_k) M(q)\) is along \(\nabla_q f(q)\) (see chapter 3, section 3.4). Hence for any vector of virtual displacements \(\delta q\) tangent to \(f(q)\), one has \(\sigma(t_k)^T M(q(t_k)) \delta q = 0\).
Example 6.2 Consider again the system of example 6.1, depicted in figure 6.1. Then we get
\[ T = \left( \frac{m_2}{\sqrt{m_1+m_2}}, -\frac{m_1}{\sqrt{m_1+m_2}} \right) \quad \text{and} \quad n_i = \left( \frac{m_2}{\sqrt{m_1+m_2}}, 1 \right). \]
It follows that
\[ T = \frac{\dot{x}_0}{m_1 + m_2} (m_2, -m_1) \quad (6.14) \]
and
\[ \ddot{x}_n = \dot{x}_n - \frac{m_2}{m_1 + m_2} \dot{x}_0 - \frac{m_1}{m_1 + m_2} \dot{x}_0 \quad (6.15) \]
Assuming \( \dot{x}_n(t_k^-) = -e\dot{x}_n(t_k^-) \) we obtain \( \dot{x}_0(t_k^-) = -e\dot{x}_0(t_k^-) \) (i.e. exactly Newton's rule), and from \( \dot{x}_n(t_k^+) = \dot{x}_n(t_k^-) \) we deduce that \( \sigma_{x_1}(t_k) = -\frac{m_2}{m_1 + m_2} \sigma_{x_0}(t_k) \), whereas \( \sigma_{x_2}(t_k) = -\frac{m_1}{m_1 + m_2} \sigma_{x_0}(t_k) \). This in particular implies that \( m_1 \sigma_{x_1} + m_2 \sigma_{x_2} = 0 \), which is the so-called conservation of linear momentum. From these equations we can calculate \( p_x = M_x \sigma_x \), where \( M_x = \text{diag}(m_1, m_2) \), or \( p_x = J^T p_x \), where \( J \) is the transformation matrix between \( x \) and \( \dot{x} \).

Remark 6.5 In the case when \( q^T = [x, \xi] \in R^6 \) where \( \xi \) represents the orientation (of a body or of the end-effector in case of a manipulator), then one can use a basis of the tangent space to the configuration space that contains the instantaneous angular velocity \( \Omega = J\dot{\xi} \) for some generally non-integrable \( J_\xi(\xi) \) (see section 4.1), and the corresponding kinetic metric. In other words, if \( T = \frac{1}{2} [\dot{\xi}^T, \Omega^T] M_\Omega \begin{bmatrix} \dot{x} \\ \Omega \end{bmatrix} \), then one can define \( n_\Omega \) using \( M_\Omega = J^T M(q) J \), where \( \dot{q} = J \begin{bmatrix} \dot{x} \\ \Omega \end{bmatrix} \). Then it is clear that some components of \( \Omega \) may possess discontinuities and obey Newton's rule for angular velocities (Through the use of an angular restitution coefficient [54] [58] [211], see (4.44)). However if Newton's rule is applied directly in a basis containing \( \dot{x} \) and the metric \( M_\xi \), then it is not obvious why this should imply a simple relationship for \( \Omega \), since \( n_\Omega \neq n_\xi \) in general.

Remark 6.6 In the case when \( f \in R^m \) \( m > 1 \), we can define \( m \) vectors \( n_{q,i} \) corresponding to \( \nabla_q f(q)_i \), and the \( n_{q,i} \)'s are independent as long as the \( \nabla_q f(q)_i \)'s are. In this case \( n_q \) is an \( n \times m \) full-rank matrix. Note that the \( n_{q,i} \)'s are not defined with the components of the matrix \( M^{-1}(q) \nabla_q f(q) \), but with \( M^{-1}(q) \nabla_q f_i(q) \) to preserve orthogonality with respect to each surface \( f_i(q) = 0 \). Also \( q_{n,i} = q_{\text{norm},i} n_{q,i} \), with \( q_{\text{norm},i} = \hat{q}^T M(q) n_{q,i} \). However it is worth noting a great difference between the case of unilateral constraints and the case when all the constraints are lasting (for instance treated in [324] or [606]). For unilateral constraints, what is the meaning of defining several normal directions? Can we associate a restitution coefficient to each of them? Roughly speaking, if the normal is not uniquely defined at an impact time, it means that the system reaches several surfaces at the same time, and that at this point they do not possess common normal directions. In other terms the system attains a singularity of the domain of constraints in the configuration space. This represents in fact the main challenge of impact dynamics, and we shall discuss this.
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in more details in section 6.4. Recall that the sweeping process formulation handles this case with a specific rule. The normal direction is replaced by the normal cone \( N(q) \) in definition 5.2. But when \( m > 1 \), existential results have been obtained only in [417], with \( T_L = 0 \).

Remark 6.7 Problem related to orthogonality have been discussed [134] in the context of hybrid force/position control of robot manipulators, where the underlying concept is *reciprocity* between twists and wrenches of the end-effector at the contact point: reciprocity means "orthogonality" between elements of the tangent space to the configuration space of the system (i.e. twists), and elements of its dual (i.e. wrenches). The main complain about "orthogonality" as usually understood in hybrid force/position control is that it is not coordinate-independent [134], and in fact since there is no scalar product in screws spaces, it is a totally meaningless notion. Note however that it might be admitted that orthogonality is basically a concept defined only between elements of dual spaces, thus closing a semantic debate. It is note worthy that a similar problem arises in percussion dynamics. The underlying concept behind the kinetic energy metric choice is duality between linear and angular momenta (i.e. \( M_\theta \dot{\theta} \) in generalized coordinates for a multibody system), and linear and angular velocities respectively [16]. Note that with friction-less constraints, the interaction force vector is always colinear to \( \nabla_X f(X) \), in any coordinates. This justifies the fact that the normal space and the tangential space are of different natures: the first one is the dual space of the second one. Thus we see that the discontinuous part of the velocity is not necessarily colinear to the percussion force vector in general coordinates.

6.3 Simple generalized impacts

6.3.1 2-dimensional lamina striking a plane

Let us consider the classical benchmark planar example of a laminar striking a fixed rigid plane (see figure 6.2). The center of gravity coordinates in the cartesian frame are given by \((x, y)\), and the orientation is \( \theta \). We define the coordinates of the point \( P \) on the boundary of the lamina closest to the constraint by \((a, b)\). The constraint is given by \( b \geq 0 \), and we suppose that \( a = x - f(\theta) \), \( b = y - h(\theta) \), where \( f(\theta) \) and \( h(\theta) \) depend on the lamina geometry. In the following, to avoid confusion between the unilateral constraint \( f(q) \geq 0 \) and the function \( f(\theta) \), we shall always write the corresponding argument. We also have to define the coordinates of the point \( P' \) that belongs to the lamina and that coincides with \( P \), whose coordinates are given by \((a', b')\). Notice that \( a'(t) = a'(0) + x(t) - x(0) + \int_0^t h(\theta) \dot{\theta} \, dt \) and \( b'(t) = b'(0) + y(t) - y(0) - \int_0^t f(\theta) \dot{\theta} \, dt \). In the following we assume that \( f(\theta) \) and \( h(\theta) \) are differentiable. Note that this may not always be the case for some body's shapes, as we shall see in the next example.

The interaction force performs work on the displacement of \( P' \). The Jacobian
between $(\dot{a}', \dot{b}')$ and $\dot{q}^T = (\dot{x}, \dot{y}, \dot{\theta})$ is given by

$$J = \begin{pmatrix} 1 & 0 & h(\theta) \\ 0 & 1 & -f(\theta) \end{pmatrix}$$

(6.16)

We can thus define two vectors of generalized coordinates $z^T = (a, b, \theta)$ and $q^T = (x, y, \theta)$. Notice that

$$p_q = J^T \begin{pmatrix} 0 \\ p_i \end{pmatrix} = \begin{pmatrix} 0 \\ p_i \\ -f(\theta)p_i \end{pmatrix}$$

(6.17)

The full-rank Jacobian between $\dot{z}$ and $\dot{q}$ is given by $J = \begin{pmatrix} 1 & 0 & F(\theta) \\ 0 & 1 & H(\theta) \end{pmatrix}$, where

$$F(\theta) = -\frac{\partial f(\theta)}{\partial \theta}, \quad H(\theta) = -\frac{\partial h(\theta)}{\partial \theta}.$$ Thus $p_z = J^{-T} p_q = \begin{pmatrix} 0 \\ p_i \\ -(H(\theta) + f(\theta))p_i \end{pmatrix} = \begin{pmatrix} 0 \\ p_i \\ 0 \end{pmatrix}$.

The last equality comes from the fact that $f(\theta) = -H(\theta)$. This kinematic relationship is true since by assumption there is no friction, hence the virtual work principle implies that the reaction is along the Euclidean normal [379]. This can easily be verified analytically in simple cases as disks or ellipses. The inertia matrix in $q$ coordinates is $M(q) = \text{diag}(m, m, I)$, where $I$ is the lamina inertia momentum. Then the inertia matrix in $z$ coordinates is given by

$$J^{-T} M(q) J^{-1} = \begin{pmatrix} m & 0 & -mF(\theta) \\ 0 & m & -mH(\theta) \\ -mF(\theta) & -mH(\theta) & mF^2(\theta) + mH^2(\theta) + I \end{pmatrix}$$

(6.18)

We assume for the moment that the surface $b = 0$ is frictionless. Thus the interaction impulse is along the surface euclidean normal, i.e. in the Cartesian 2-dimensional space $p = \begin{pmatrix} 0 \\ p_i \end{pmatrix}$. Applying the general relationship in (6.1), it follows that

$$m \sigma_a(t_k) - mF(\theta) \sigma_\theta(t_k) = 0$$

(6.19)
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\[ m\sigma_\beta(t_k) - mH(\theta)\sigma_\beta(t_k) = p_i \]  
\[ -mF(\theta)\sigma_\delta(t_k) - mH(\theta)\sigma_\delta(t_k) + (mF^2(\theta) + mH^2(\theta) + I)s_\delta(t_k) = 0 \]

We shall proceed as follows: a) Firstly we apply a complete Newton’s rule using the Euclidean metric. We show that this makes no sense, because the obtained results in new coordinates do not correspond to an equivalent restitution rule, and the obtained results together with the dynamical shock equations yield a contradiction. b) Secondly we apply an intuitive rule about continuity of some tangential velocity\(^4\), and we also exhibit some inconsistencies. c) Thirdly we apply Newton’s rule to the Euclidean normal components and we calculate the rest of the postimpact velocities \textit{via} the dynamical equations. This is equivalent to apply a complete Newton’s rule with the kinetic metric. Then we use the kinetic metric as described above.

a) Suppose that we apply a complete Newton’s restitution rule in \(x\) coordinates, along the euclidean normal to the constraint \(b = 0\), i.e. 
\[
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\] 
This means that we assume that \(\sigma_\beta(t_k) = \sigma_\delta(t_k) = 0\). Then it follows from (6.19) (6.20) (6.21) that \(m\sigma_\beta(t_k) = p_i\), and \(p_i\) can be computed from \(\dot{b}(t_k^+) = -eb(t_k^-)\). Thus the restitution matrix is given by \(E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -e & 0 \\ 0 & 0 & 1 \end{pmatrix}\). Simple calculations using the Jacobian \(J\) above show that in \(q\) coordinates, one does not obtain a complete Newton’s rule, but the following relationships

\[ \dot{x}(t_k^+) = \dot{x}(t_k^-) \]  
\[ \dot{y}(t_k^+) = -e\dot{y}(t_k^-) - (1 + e)H(\theta)\dot{\theta}(t_k^-) \]  
\[ \dot{\theta}(t_k^+) = \dot{\theta}(t_k^-) \]

Since the normal to the constraint is now given by \(\nabla_q f(q) = \begin{pmatrix} 0 \\ 1 \\ H(\theta) \end{pmatrix}\), there is no correspondance with any simple restitution rule in these coordinates. We should get \(\sigma_\beta(t_k) \neq 0\), which is not the case in general in (6.24). \(\sigma_\delta(t_k) \neq 0\) can be true if \(H(\theta) = 0\), which is the case for instance if the lamina is a disk, since in that case \(h(\theta) = R\). Now note that (6.19) (6.20) (6.21) can be used to compute that \(H(\theta)\sigma_\beta(t_k) = 0\), which together with the restitution rule yields a contradiction except if \(H(\theta) = 0\) at the impact time (that is the case for a disk). Notice that it suffices that locally around the contact point the lamina’s center of gravity be such that \(F(\theta) = H(\theta) = 0\): it may happen that for some bodies with a nonhomogeneous repartition of mass, these equalities are verified without the geometry being that of a disk.

\(^4\)which in fact neglects inertial effects that may sometimes induce tangential velocity jumps, see (4.25) for the two bodies 3-dimensional case.
b) Let us now assume\(^5\) that \(\sigma_\theta(t_k) = 0\), because there is no friction and thus there is no tangential impulse, while a restitution rule is applied to \(\dot{b}\). Then from (6.19) (6.20) (6.21) we get

\[
\sigma_\theta(t_k) = \frac{H(\theta)}{mF^2(\theta) + mH^2(\theta) + I} \sigma_\theta(t_k)
\]  

(6.25)

Now notice that from (6.19) it follows that \(a_\theta(t_k) = 0\), except if \(F(\theta) = 0\). Thus from (6.25) we get \(a_\theta = 0\), which thus constitutes a contradiction in general when \(F(\theta) \neq 0\). If we do not a priori suppose that \(\dot{a}\) is continuous, then we get

\[
\sigma_\theta(t_k) = \frac{-mF(\theta)f(\theta)}{I + mf^2(\theta)} \sigma_\theta(t_k)
\]  

(6.26)

and

\[
\sigma_\theta(t_k) = \frac{-mf(\theta)}{I + mf^2(\theta)} \sigma_\theta(t_k)
\]  

(6.27)

One therefore realizes that there is in general effectively a discontinuity in the tangential velocity due to inertia coupling. Recall that this is calculated from the assumption that we apply Newton’s rule to \(b\), and then we compute the other velocity changes from the dynamical shock equations.

c) Note that it has been possible to obtain all the postimpact velocities in terms of the preimpact ones using only \(\dot{b}(t_k^-) = -eb(t_k^-)\) and the dynamical equations in (6.19) (6.20) (6.21). The same result would have been obtained working with the transformed coordinates in (6.6). We get for the \(q\) coordinates:

\[
\dot{y}(t_k^+) + H(\theta)\dot{\theta}(t_k^+) = -e \left( \dot{y}(t_k^-) + H(\theta)\dot{\theta}(t_k^-) \right)
\]  

(6.28)

We have:

\[
n_q = \sqrt{\frac{mI}{mH^2(\theta) + I}} \begin{pmatrix} 0 \\ \frac{1}{m} \\ \frac{H}{I} \end{pmatrix}
\]  

(6.29)

Then defining two tangential vectors \(t_{q,1}\) and \(t_{q,2}\) as follows:

\[
t_{q,1} = \begin{pmatrix} \frac{1}{\sqrt{m}} \\ 0 \\ 0 \end{pmatrix} \quad t_{q,2} = \frac{1}{\sqrt{mH^2(\theta) + I}} \begin{pmatrix} 0 \\ -H(\theta) \\ 1 \end{pmatrix}
\]  

(6.30)

we get \(\sigma_\pm = 0\) and:

\[
\dot{\theta}(t_k^+) = \frac{1}{mH^2(\theta) + I} \left[ -mH(\theta)(1 + e)\dot{y}(t_k^-) + (I - mH^2(\theta)e)\dot{\theta}(t_k^-) \right]
\]  

(6.31)

\(^5\)Following for instance [574], whose dynamical equations correspond in fact to the choice of a vector of generalized coordinates \(\bar{q} = \begin{pmatrix} x \\ b \\ \theta \end{pmatrix}\).
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\[ \dot{y}(t_k^+) = \frac{1}{mH^2(\theta) + I} \left[ (mH^2(\theta) - eI)\dot{y}(t_k^-) - IH(\theta)(1 + e)\dot{\theta}(t_k^-) \right] \quad (6.32) \]

and the kinetic energy loss is given by

\[ T_L = (e^2 - 1)\frac{mI}{mH^2(\theta) + I} (\dot{y}(t_k^-) + H(\theta)\dot{\theta}(t_k^-))^2 \quad (6.33) \]

In this case we have

\[ \dot{q}_{\text{norm}} = \sqrt{\frac{mI}{mH^2(\theta) + I}} (\dot{y} + H(\theta)\dot{\theta}) \quad (6.34) \]

and

\[ \dot{q}_{\text{tang}} = \left( \frac{\dot{x}}{\sqrt{mH^2(\theta) + I}} \right) \quad (6.35) \]

It can be checked that (6.26) (6.27) and (6.31) (6.32) are equivalent.

To conclude this part, we can state the following

Claim 6.1 Consider a Lagrangian system with generalized coordinates \( q \in \mathbb{R}^n \), submitted to a frictionless unilateral constraint \( f(q) \geq 0 \) of codimension 1. Then it is equivalent to

i) Apply Newton’s restitution rule to the component of \( \dot{q} \) along \( \nabla_q f(q) \) (i.e. \( \dot{q}^T \nabla_q f(q) = -e\dot{q}^T \nabla_q f(q) \)), and then calculate the remaining postimpact velocity components using the percussion dynamics \( M(q)\sigma_q = p_q, p_q \in \text{span} \nabla_q f(q) \)

or to

ii) Apply a complete Newton’s restitution rule using the kinetic metric, i.e. set \( \dot{q}_n(t_k^+) = -e\dot{q}_n(t_k^-), \dot{q}_t(t_k^+) = \dot{q}_t(t_k^-) \), where \( \dot{q}_n = \dot{q}^T M(q) n_q n_q, n_q = \frac{M^{-1}(q)\nabla_q f(q)}{\sqrt{\nabla_q f(q)^T M^{-1}(q)\nabla_q f(q)}}, \dot{q}_t = \dot{q} - \dot{q}_n \).

In summary, the dynamical equations in (6.6) show that applying a generalized restitution rule to \( \dot{q}_{\text{norm}} \) or to \( \dot{q}^T \nabla_q f(q) \) is the only possible solution (not in the sense that the restitution coefficient are the only possible models, see remark 4.5, but in the sense that all the rest of the velocity remains continuous at impacts). Hence in such transformed velocities the restitution operator \( \mathcal{E} \) has a quite simple form

\[ \mathcal{E} = \begin{pmatrix} -e & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (6.36) \]

In the failing lamina case, the restitution operator takes on the form

\[ \mathcal{E}_q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -e & 0 \\ 0 & \frac{(1+e)mf(\theta)}{I+mf^2(\theta)} & 1 \end{pmatrix} \quad (6.37) \]
in $\tilde{q}$-coordinates, and

$$
\mathcal{E}_q = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{mH^2(\theta) - eI}{mH^2(\theta) + I} & -I \frac{H(\theta)(1+e)}{mH^2(\theta) + I} \\
0 & \frac{-mH(\theta)(1+e)}{mH^2 + I} & \frac{I - mH(\theta)e}{mH^2(\theta) + I}
\end{pmatrix}
$$

(6.38)

in $q$-coordinates, and finally

$$
\mathcal{E}_z = \begin{pmatrix}
1 & \frac{(1+e)mF(\theta)f(\theta)}{I + mF^2(\theta)} & 0 \\
0 & -e & 0 \\
0 & \frac{mf(\theta)(1+e)}{I + mF^2(\theta)} & 1
\end{pmatrix}
$$

(6.39)

in $z$-coordinates. What is important to recall is that whatever the coordinates $\rho$ one may use, and the corresponding restitution matrix, then the discontinuous part of the velocity is the corresponding $\dot{\rho}_{\text{norm}}$, and Newton's rule applies to $\dot{\rho}(t_k)^T \nabla_{\rho} f(\rho)$. Also one can check that the kinetic energy loss can be calculated in any coordinate system. For instance when $e = 1$ then $\mathcal{E}_q^T M(q) \mathcal{E}_q = M(q)$ for any coordinates $q$.

**Remark 6.8** Note in relationship with claim 4.22 that we can write

$$
\bar{J}^T \mathbf{n} = \nabla_q f(q)
$$

(6.40)

where $\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the normal vector to the rigid base, $\bar{J} = E_2 \mathcal{M}$, $\mathcal{M} = \begin{pmatrix}
1 & 0 & h(\theta) \\
0 & 1 & -f(\theta) \\
0 & 0 & 1
\end{pmatrix}$, $\mathcal{M} \dot{\mathbf{q}} = \begin{pmatrix} \ddot{\mathbf{a}} \\ \ddot{\mathbf{b}} \\ \dot{\theta} \end{pmatrix}$, $E_2 = [I_2 : 0] \in \mathbb{R}_{2 \times 3}$. Moreover the application of (4.40) yields

$$
-e = \frac{V_F(t_k^+)^T \mathbf{n}}{V_P(t_k^+)^T \mathbf{n}} = \frac{\dot{y}(t_k^+) + H(\theta)\dot{\theta}(t_k^+)}{\dot{y}(t_k^+) + H(\theta)\dot{\theta}(t_k^+)}
$$

(6.41)

which is exactly the same result as if one applies directly $\dot{\mathbf{b}}(t_k^+) - e \dot{\mathbf{b}}(t_k^+)$, or $\dot{q}_{\text{norm}}(t_k^+) = -e \dot{q}_{\text{norm}}(t_k^+)$. However it is clear, as we pointed out at the beginning of this section, that stating continuity of the tangential part of the velocity, i.e. $V_P^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \dot{x} + h(\theta)\dot{\theta}$ would: firstly yield a wrong result (check that $\dot{x}(t_k^+) + h(\theta)\dot{\theta}(t_k^+) \neq \dot{x}(t_k^+) + h(\theta)\dot{\theta}(t_k^+)$ from the continuity of $\dot{q}_{\text{tang}}$). Secondly it is not sufficient to solve the impact problem (it provides 3 equations for 4 unknowns).

**Remark 6.9** Time-varying unilateral constraints From now on, we have considered only classical constraints of the form $f(q) \geq 0$. What about the case
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\[ f(q, t) \geq 0? \text{ Which restitution rule should we apply? If } f(q, t) \text{ is smooth enough with respect to } t, \text{ then one may think at first sight to apply} \]

\[ \dot{q}(t_k^+)^T \nabla_q f(q, t) = -e \dot{q}(t_k^-)^T \nabla_q f(q, t) \]  \hspace{1cm} (6.42)

\[ \text{since at a given instant, the normal direction is } \nabla_q f(q, t). \text{ However there is then no difference with the case when the constraints are not moving. It seems however intuitively that the constraint velocity at the impact time should be taken into account, since the constraint will transmit some energy to the system that strikes it. This can be done as follows, by defining an extended restitution rule (may be a dynamic restitution rule) as} \]

\[ (\dot{q}(t_k^+)^T, 1) \left( \begin{array}{c} \nabla_q f(q, t) \\ \nabla_t f(q, t) \end{array} \right) = -e(\dot{q}(t_k^-)^T, 1) \left( \begin{array}{c} \nabla_q f(q, t) \\ \nabla_t f(q, t) \end{array} \right) \]  \hspace{1cm} (6.43)

\[ \text{i.e.} \]

\[ \dot{q}(t_k^+)^T \nabla_q f(q, t) = -e \dot{q}(t_k^-)^T \nabla_q f(q, t) - (1 + e) \nabla_t f(q, t) \]  \hspace{1cm} (6.44)

\[ \text{In this case the velocity of the constraint clearly plays a role in the calculation of the postimpact system's velocity. This extended dynamic restitution rule is motivated by the following example. Consider two masses } m_1, m_2 (\text{this may represent a bouncing ball on a table, or a juggling one degree-of-freedom robot), moving vertically} \]

\[ \begin{align*}
\dot{x}_1(t) &= -m_1 g \\
\dot{x}_2(t) &= u - m_2 g \\
x_1(t) - x_2(t) &\geq 0
\end{align*} \]  \hspace{1cm} (6.45)

\[ \text{Newton's rule gives } \dot{x}_1(t_k^+) - \dot{x}_2(t_k^+) = -e(\dot{x}_1(t_k^-) - \dot{x}_2(t_k^-)) \text{ at the impact times. Now under the assumption that } m_2 >> m_1, \dot{x}_2 \text{ remains time-continuous, and we may consider the second mass as representing a time-varying unilateral constraint } x_2(t), \text{ i.e. we now deal with the reduced order system} \]

\[ \begin{align*}
\dot{x}_1(t) &= -m_1 g \\
x_1(t) - x_2(t) &\geq 0
\end{align*} \]  \hspace{1cm} (6.46)

\[ \text{Then the restitution rule becomes } \dot{x}_1(t_k^+) = -e \dot{x}_1(t_k^-) + (1 + e) \dot{x}_2(t). \text{ Notice that if we applied the rule in (6.42) we would obtain } \dot{x}_1(t_k^+) = -e \dot{x}_1(t_k^-), \text{ but applying (6.43) we just obtain (6.46). It is clear that one cannot neglect in the postimpact velocity calculation the velocity of mass 2. When this mass is to be considered as a moving constraint, then its velocity at the collision time must be taken into account.} \]

\textbf{Remark 6.10} From (6.19) (6.20) (6.21) one sees that we can write an equation like

\[ \mathcal{M}(\theta) \left( \begin{array}{c} \sigma_a(t_k) \\ \sigma_b(t_k) \\ p_i \end{array} \right) = \mathcal{H}(\theta) \sigma_b(t_k) \]  \hspace{1cm} (6.47)
for some suitable matrices $\mathcal{M}(\theta)$ and $\mathcal{H}(\theta)$. It happens that $\mathcal{M}(\theta)$ is full-rank in that case, making the impact problem solvable.

**Remark 6.11 On manipulator feedback control**

In remark 6.1 we have outlined how the transformed velocities $\dot{q}_{\text{norm}}$ and $\dot{q}_{\text{ang}}$ could be used to design a stable closed-loop transition phase for a rigid manipulator colliding a rigid obstacle. Let us now consider the transformation proposed in [324] that is convenient for hybrid force/position control of manipulators with a set of independent holonomic constraints $f(q) = 0$, $f \in \mathbb{R}^m$. Roughly, it is assumed that the generalized coordinates vector $q \in \mathbb{R}^n$ can be rearranged in such a way that for some $q_2 \in \mathbb{R}^{n-m}$, $f(q_2), q_2) = 0$. The existence of $q_2$ can be proved with the implicit function theorem, and is assumed to hold globally. Then the following coordinate change is considered

$$x = X(q) = \begin{bmatrix} q_1 - \Omega(q_2) \\ q_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(6.48)

with inverse transformation

$$q = Q(x) = \begin{bmatrix} x_1 + \Omega(x_2) \\ x_2 \end{bmatrix}$$

(6.49)

and full-rank Jacobian $J_x$. The constraints become simply $x_1 = 0$ in $x$-coordinates. Then the dynamical equations of the rigid manipulator are transformed into

$$J^T_x M J_x \ddot{x} + J^T_x H(x, \dot{x}) = J^T_x u + J^T_x f_e$$

(6.50)

where $H(x, \dot{x})$ denotes the Coriolis and centrifugal terms, $u$ is some external generalized action (for instance joint torques as inputs), and $f_e$ is the constraint reaction. Now define $I_1 = [E_1^T : E_2^T]$, $E_1 \in \mathbb{R}^{m \times n}$, $E_2 \in \mathbb{R}^{(n-m) \times n}$, and for all $y \in \mathbb{R}^n$ one gets $y = \begin{bmatrix} E_1 y \\ E_2 y \end{bmatrix}$. It follows that the dynamics in (6.50) can be split into two parts as:

$$\begin{cases} E_1 J^T_x M J_x \ddot{x} + E_1 J^T_x H(x, \dot{x}) = E_1 J^T_x u + E_1 J^T_x f_e \\ E_2 J^T_x M J_x \ddot{x} + E_2 J^T_x H(x, \dot{x}) = E_2 J^T_x u \end{cases}$$

(6.51)

Note that due to the absence of friction, $J^T_x f_e = \sum_{k=1}^m \lambda_k \nabla x f_k(x) = \sum_{k=1}^m \lambda_k e_k$, where $e_k$ is the $k$-th unit vector of $\mathbb{R}^n$, so that $E_2 J^T_x f = 0$. Let us now assume that $m = 1$. This is due to the fact that, as we have seen, assuming that several constraints are attained simultaneously yields much complications for the definition of the impact law. Contrarily to what is done in [324], we cannot use here the fact that $x_1 = 0$, because the system rebounds on the surface, but is not in permanent constraint state. It could be stated that only $\dot{x}_1$ possesses discontinuities, because $x_2$ characterizes the tangential motion along the constraint. But note that if such is
the case, then the equations in (6.51) will in general yield a contradiction because 
one gets at the impact time

$$E_2J_x^T MJ_x \begin{pmatrix} \sigma_{x_1} \\ 0 \end{pmatrix} = 0_{n \times 1}$$  \hspace{1cm} (6.52)

Hence except if the vector $J_x^T MJ_x \begin{pmatrix} \sigma_{x_1} \\ 0 \end{pmatrix}$ lies in the kernel of $E_2$, this is impossible.

This is the problem of applying a generalized restitution rule that we discussed on
the lamina example. Now let us write $x = \begin{pmatrix} x_1 \\ 0 \\ \ddots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$. Is the vector $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$
colinear to $\nabla_x f_{eq}(x)$? The answer is positive, since in the transformed coordinates $x$ the constraint is simply $x_1 = 0$. In other words, the system is submitted to the unilateral constraint $x_1 \geq 0$, and the Euclidean normal direction is simply $\nabla_x x_1^T = (1, 0, ..., 0)$. Then we can apply Newton's restitution rule to $\dot{x}_1$ and calculate the rest of the velocities jumps with the dynamics, using the second equation in (6.51) at the impact time. More precisely, if $C_i$, $1 \leq i \leq n$ denote the columns of the matrix $J_x^T MJ_x$, then the second equation in (6.51) yields

$$\sigma_{x_2} = - \left( E_2 [C_1; \ldots; C_n] \right)^{-1} E_2 C_1 \sigma_{x_1}$$  \hspace{1cm} (6.53)

where we have used the fact that the $(n-1) \times (n-1)$ matrix $E_2 [C_1; \ldots; C_n]$ is full-rank (It is an $(n-1) \times (n-1)$ minor of the inertia matrix). The percussion can be computed then using the first equation in (6.51). But now, what about the design of a stabilizing controller during an impact phase? (We assume here that the goal is to stabilize in finite time the manipulator's tip on the surface). One can apply a linearizing controller to get a first-stage closed-loop equation $\ddot{x} = v$ (see (6.50)). The first component of $v$ can be set equal to -1, so that the dynamics for the coordinate $x_1$ is that of the bouncing ball

$$\ddot{x}_1 = -1 \quad x_1 \geq 0$$  \hspace{1cm} (6.54)

and $x_1(t)$ tends towards zero in finite time. For the rest of the velocities, i.e. $\dot{x}_2$, we are left with an initial value problem after each impact, i.e.

$$\begin{cases} 
\dot{x}_2 = v_2 \\
\dot{x}_2(t_k(t_k^-)) = \dot{x}_{2,k} \\
x_2(t_k) = x_{2,k}
\end{cases}$$  \hspace{1cm} (6.55)
It does not seem obvious to make $x_2$ and $\dot{x}_2$ track some desired trajectories. The obtained $x_2$-subsystem is typically a smooth system with exogeneous jump conditions. A possible path is to investigate how the "control" that appears in these jumps (i.e. in fact $\dot{x}_1$, see (6.53), and hence indirectly $v_1$) can be designed in accordance with $v_2$ to get a desired motion during the impact phase.

Notice in relation with remark 2.5 that from (6.54) and (6.55) the generalized velocity is of local bounded variation, from the choice of the decoupling and linearizing state feedback law. Nevertheless the work in [416] could be useful when such decoupling and linearization is not performed accurately, due to application of another type of controller, or due to bad knowledge of the system’s parameters. For instance in chapter 8 we shall see the application of switching controllers. Then more elaborate tools will have to be used to prove that the velocity is of local bounded variation.

At this stage, it is therefore not really clear yet which kind of coordinate transformation should be used to design a transition phase control law from free to constrained motion that guarantees at the same time: stabilization in the normal direction to the surface in finite time, and tracking of a reference trajectory in the "tangential" direction.

Remark 6.12 Assume the surface of constraints has codimension one. Then as shown in [85] there exists a generalized coordinates change such that the normal vector to the hyperplane $f(q) = 0$ in the corresponding kinetic metric is proportional to the Euclidean normal vector $\nabla_qf(q)$ for some coordinates $\tilde{q}$, i.e. $n_q = \alpha \nabla_qf(q)$ for some real $\alpha$. This coordinate change is given in two steps:

\[
q = (q_1, \ldots, q_n) \rightarrow \tilde{q} = (q_1, \ldots, q_{n-1}, f(q)) \rightarrow \tilde{q} = q - q_n \left( \frac{\tilde{M}^{-1}\mathcal{N}}{\tilde{N}^T \tilde{M}^{-1}\mathcal{N}} - \mathcal{N} \right) \quad (6.56)
\]

where $\mathcal{N} = \nabla_qf_0(q) = (0, \ldots, 0, 1)$. The coordinates $\tilde{q}$ are the so-called quasi-coordinates of the system [420]. A nice feature of the new coordinates $\tilde{q}$ is that the inertia matrix becomes then quite simple since $\tilde{m}_{1i}(\tilde{q}) = 0$ for $i = 1, \ldots, n - 1$. This holds locally around $\tilde{q}_n = 0$. As an example, consider the lamina striking a horizontal plane. Then $\tilde{q} = \begin{pmatrix} x \\ y \\ y - h(\theta) \end{pmatrix}$. It follows that in $\tilde{q}$-coordinates, the inertia matrix is given by

\[
\tilde{M} = \begin{pmatrix}
m & 0 & 0 \\
0 & \frac{I + mH^2(\theta)}{H^2(\theta)} & -\frac{I}{H^2(\theta)} \\
0 & -\frac{I}{H^2(\theta)} & \frac{I}{H^2(\theta)} \end{pmatrix} \quad (6.57)
\]
One computes that $\ddot{q} = \left( \ddot{q}_2 - \ddot{q}_3 \frac{I}{I + mH^2(\theta)} \right) = J(\theta) \ddot{q}$. Now locally around $\theta = 0$, we have $\dot{q}_3 = J(\theta) \dot{q}_3$ since $J(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{I}{mH^2(\theta) + I} \\ 0 & 0 & 1 \end{pmatrix}$, so that $J(\theta, \dot{\theta}) \ddot{q}_3 = 0$ for $\ddot{q}_3 = 0$. Hence we treat $J(\theta)$ as the Jacobian between $\dot{q}$ and $\ddot{q}$. Then one gets in $\dot{q}$-coordinates

$$M = \begin{pmatrix} m & 0 & 0 \\ 0 & \frac{mH^2(\theta) + I}{H^2(\theta)} & 0 \\ 0 & 0 & \frac{ml}{mH^2(\theta) + I} \end{pmatrix}$$

and it can be verified that $n_\dot{q} = \frac{M^{-1}N}{\sqrt{N^T M^{-1} N}} = N$, where $N = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. $N$ is the Euclidean normal to the constraint in these coordinates, i.e. $\ddot{q}_3 = 0$.

### 6.4 Multiple generalized impacts

In the foregoing section, we have dealt with simple impacts, i.e. collisions with a codimension one constraint. We now enlarge the study to multiple impacts. We start in the next subsection with the rocking block example, and with a simplified restitution rule with a limited domain of applications. Motivated by this example, we search for more sophisticated models in subsection 6.5.

#### 6.4.1 The rocking block problem

The problem concerning the modelization of the rolling (or rocking) block has attracted the attention of numerous researchers. The reasons are that on one side this is a challenging problem concerning the definition of suitable restitution rules that allow to describe the block motion, and that it has important applications for the understanding of buildings (slender water tanks, petroleum cracking towers [212], ancient Greek and Roman stone temples [157]) motion during earthquakes, as well as stacks of graphite blocks in the core of nuclear reactors [406]. Basically, a slender rigid block, moving on a fixed base, that rotates around one of its corners (say $A$), possesses the same dynamical equations as an inverted pendulum [213], i.e.

$$I \ddot{\theta} = -g \frac{\sqrt{L^2 + l^2}}{2} \cos(\theta + \alpha)$$
where the various terms in (6.59) are defined on figure 6.3, and $\alpha = \arctan(\frac{L}{l})$. It is noteworthy that (6.59) is valid when the center of rotation is A. When the block rotates around B, the gravity torque becomes equal to $g\frac{\sqrt{l^2 + l^2}}{2} \cos(-\theta + \alpha)$. However, as we are going to see in this subsection, the main difference between the inverted pendulum and the rocking block dynamics lies in the unilateral constraints that exist for the latter.

Let us consider a rectangular block falling on a plane as in figure 6.3 (this is known as the SRM simple rocking model in the related literature). We have $f(\theta) = \frac{L}{2} \cos(\theta) - \frac{L}{2} \sin(\theta)$ outside $\theta = 0$ and when A is lower than B, and $f(\theta) = -\frac{L}{2} \cos(\theta) + \frac{L}{2} \sin(\theta)$ when B is lower than A. Also $2h(\theta) = l \cos(\theta) + L \sin(\theta)$ for point A and $2h(\theta) = l \cos(\theta) - L \sin(\theta)$ for point B. L and l are the lengths of the edges. Then the point closest to the plane is one of the vertices of the downward edge of the block. Thus $|f(\theta = 0)| = \frac{L}{2}$. But $f(\theta = 0)(t^+_{\theta}) = \frac{L}{2}$ and $f(\theta = 0)(t^-_{\theta}) = -\frac{L}{2}$. In this case $f(\theta)$ is discontinuous at $0$. Even more, it is not defined at $\theta = 0$, since the whole edge is at the same distance to the plane. Then should one choose a particular point of the edge to define $f(0)$, or should one state that $f(\theta = 0) \subseteq [-\frac{L}{2}, \frac{L}{2}]$? What are the consequences of such a choice on the impact-contact law that may be chosen to represent the percussion, on the system's dynamics? A way to avoid this problem is to proceed as follows: when the block is close enough to the plane, there is an additional constraint that states that the rotation $\theta$ is limited between two bounds that depend on $y$, namely

$$\theta \in [-\theta_M(y), \theta_M(y)]$$

(6.60)

with

$$\theta_M(y) = \arcsin\left(\frac{2y}{\sqrt{l^2 + L^2}}\right) - \arctan\left(\frac{l}{L}\right)$$

(6.61)

where we assume also that $\theta + \arctan(\frac{L}{l}) \leq \frac{\pi}{2}$. Recall that arcsin is the inverse function of sine, defined from $[-1, +1]$ into $[-\frac{\pi}{2}, \frac{\pi}{2}]$, i.e. $\sin(\cdot) \circ \arcsin(\cdot)$ is the identity function, and similarly for $\tan(\cdot)$ where $\arctan(\cdot)$ is defined from $\mathbb{R}$ into $[-\frac{\pi}{2}, \frac{\pi}{2}]$. 

Figure 6.3: Rigid planar rectangular block (Simple Rocking Model).
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Notice that \( \theta_M(\frac{1}{2}) = 0, \theta_M(\sqrt{\frac{L^2 + L^2}{2}}) = \frac{\pi}{2} - \arctan(\frac{1}{L}) \). Assume we incorporate this additional bi-unilateral constraint into the problem. Then can we associate to this constraint an angular restitution coefficient? Would this solve the problem when one vertex is in contact and the block rotates around it until it collides the plane with the whole edge? What is the relationship between this and the definition of \( f(\theta) \) at 0? The goal of this example is to show what the use of the kinetic metric in the configuration space can bring to the impact problem.

Note that it is equivalent, when the constraints on \( \theta \) exist, to state that \( b'' \geq 0 \) where \( b'' = b \) if \( b < b' \), \( b'' = b' \) if \( b < b' \), and \( b'' = b = b' \) if \( b = b' \), where \( b \) and \( b' \) are the vertical coordinates of the end points of the downwards edge of the block \( A \) and \( B \), see figure 6.3. The Jacobian between \( q = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \) and \( q_A = \begin{pmatrix} a \\ b \\ \theta \end{pmatrix} \) is equal to

\[
J = \begin{pmatrix}
1 & 0 & \frac{1}{2} \cos(\theta) + \frac{L}{2} \sin(\theta) \\
0 & 1 & \frac{L}{2} \sin(\theta) - \frac{L}{2} \cos(\theta) \\
0 & 0 & 1
\end{pmatrix}.
\]

It can be as well calculated between \( q \) and \( q_B \).

Let us denote the two constraints on \( \theta \) as:

\[
\begin{cases}
f_1(q) = \theta + \theta_M(y) \geq 0 \\
f_2(q) = \theta - \theta_M(y) \leq 0
\end{cases}
\]

Note that they could be written as well as follows:

\[
\begin{cases}
f_2(q) = b' = y - \frac{1}{2} \cos(\theta) - \frac{L}{2} \sin(\theta) \geq 0 \\
f_1(q) = b = y - \frac{1}{2} \cos(\theta) - \frac{L}{2} \sin(\theta) \geq 0
\end{cases}
\]

We now study what happens when the configuration in figure 6.4 occurs, i.e. the block rotates around the left corner. In this case \( f_2(q) = 0 \) and \( f_1(q) \geq 0 \). This is therefore typically a problem where a Lagrangian system (quite simple in this case), is submitted to several unilateral constraints, some of which are lasting. We have \( M(q) = \text{diag}(m, m, I) \), with \( I = \frac{m}{12}(L^2 + L^2) \). We naturally choose to apply restitution rules using \( f_1(q) = 0 \) as the surface of impact, since the point strikes this surface. The normal direction is \( \nabla_q f_1(q) = \begin{pmatrix} 0 \\ \frac{2}{\sqrt{I^2 + L^2 - 4y^2}} \\ 1 \end{pmatrix} \), so that \( q_{\text{norm},1} = \)
\( \frac{q^T \nabla q f_1(q)}{\sqrt{\nabla q f_1(q)^T \mathcal{M}^{-1}(q) \nabla q f_1(q)}} \) and \( q^T \nabla q f_1(q) = \frac{2}{\sqrt{r^2 + L^2 - q^2}} \dot{y} + \dot{\theta} \). Note that using (6.63) one finds the same normal direction. Let us apply Newton’s restitution rule to \( \dot{q}_{\text{norm}} \). We get

\[
\frac{2}{L} \dot{y}(t^n_k) + \dot{\theta}(t^n_k) = -e_n \left( \frac{2}{L} \dot{y}(t^n_k) + \dot{\theta}(t^n_k) \right)
\]

(6.64)

because at the impact times \( \theta = 0, y = \frac{1}{2} \) and \( \frac{2}{\sqrt{r^2 + L^2 - q^2}} = \frac{2}{L} \). Now let us assume that \( p_q = \lambda_1 \nabla q f_1(q) \) for some \( \lambda_1 \in \mathbb{R}^+ \), i.e. we assume that the component of the generalized percussion along \( \nabla q f_2(q) \) is zero. Thus the dynamic equations are

\[
\begin{align*}
\begin{cases}
 m\sigma_{x} &= 0 \\
m\sigma_{y} &= \frac{2}{L} \lambda_1 \\
I\sigma_{\theta} &= \lambda_1
\end{cases}
\end{align*}
\]

(6.65)

It follows that \( \dot{x} \) remains continuous at the impact, while from (6.64) and (6.65) we obtain

\[
\begin{align*}
m\dot{y}(t^n_k) &= m\dot{y}(t^n_k) + \frac{2}{L} \lambda_1 \\
I\dot{\theta}(t^n_k) &= I\dot{\theta}(t^n_k) + \lambda_1 \\
\frac{2}{L} \dot{y}(t^n_k) + \dot{\theta}(t^n_k) &= -e_n \left( \frac{2}{L} \dot{y}(t^n_k) + \dot{\theta}(t^n_k) \right)
\end{align*}
\]

(6.66)

From (6.66) it follows that

\[
\dot{y}(t^n_k) = \frac{1}{4I + mL^2} \left[ (mL^2 - 4le_n)\dot{y}(t^n_k) + 2LI(1 + e_n)\dot{\theta}(t^n_k) \right]
\]

(6.67)

and

\[
\dot{\theta}(t^n_k) = \frac{1}{4I + mL^2} \left[ -2mL(1 + e_n)\dot{y}(t^n_k) + (4I - mL^2e_n)\dot{\theta}(t^n_k) \right]
\]

(6.68)

Notice that the expressions in (6.67) and (6.68) are independent of the mass \( m \). They are functions of the dimensions and the restitution only. Also we can further simplify them, noting that before the shock \( f_2(q) = 0 \). Hence \( \dot{y}(t^n_k) = \frac{L}{2} \dot{\theta}(t^n_k) \). It follows that

\[
\dot{y}(t^n_k) = \frac{mL^2 - 8le_n - 4I}{4I + mL^2} \dot{y}(t^n_k) = \frac{2L^2 - l^2 - 2e_n(L^2 + l^2)}{l^2 + 4L^2} \dot{y}(t^n_k)
\]

(6.69)

and

\[
\dot{\theta}(t^n_k) = -(2e_n + 1) mL^2 + 4I \dot{\theta}(t^n_k) = \frac{(2e_n + 1)(L^2 + l^2) + 4(l^2 + L^2)}{4I + 16L^2} \dot{\theta}(t^n_k)
\]

(6.70)

\(^{6}\text{Recall from the two bodies case in chapter 4, equations (4.25) and (4.44), that the percussion vector and the restitution coefficients to be defined are closely related. Here we impose a form of the percussion, and we investigate to which kind of postimpact motion it corresponds. In subsection 6.5.2 we shall examine the general case.}\)
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Figure 6.5: Singularity in the rocking block problem (Points $a$, $b$ and $c$ correspond to figure 6.6).

Now from figure 6.5 one sees that the system is constrained in the 3-dimensional configuration space by a curve (the figure represents a section in the $(y, \theta)$ plane) given by the two equations $f_1(q) = f_2(q) = 0$. This curve possesses a cusp singularity at $\theta = 0$, $y = \frac{1}{2}$, that exactly corresponds to the block making contact with the plane through a line contact. Do equations in (6.69) and (6.70) yield a feasible motion? i.e. does the chosen model (a generalized shock "against" the constraint $f_1(q) = 0$) correspond to a possible motion of the block? Looking at figure 6.5, one sees that the system (i.e. a point in the plane $(y, \theta)$) follows the curve $f_2(q) = 0$ and then strikes the curve $f_1(q) = 0$. If the point remains at the singularity, then it means that the block has stopped to move after the percussion. If the point leaves $f_2(q) = 0$ and moves along $f_1(q) = 0$, then the block has started a rotation around its right corner. This is depicted in figure 6.6. The configuration space thus allows to clearly visualize the possible motions of the block, and the singular point $\theta = 0$, $y = \frac{1}{2}$ corresponds to a sort of generalized potential well where the block should asymptotically converge provided some dissipation is present in the model (Here the only dissipation is at the impacts, since only gravity acts on the system between impacts).

However it is clear that equations (6.69) and (6.70) will not always yield a feasible motion. Indeed a feasible motion implies that $\dot{y}(t_k^+) \geq 0$, $\dot{\theta}(t_k^+) \leq 0$ if the block rotates around $A$ again, $\dot{\theta}(t_k^+) \geq 0$ if the block rotates around $B$. In particular, since $\dot{y}(t_k^-) < 0$, the coefficient in (6.69) has to be negative. In fact, a sufficient condition for that is that the tangential postimpact velocity is in the right direction along $f_1(q) = 0$. Recall that for the moment, we have $\dot{q}_{\text{ang},1}(t_k^+) = \dot{q}_{\text{ang},1}(t_k^-)$ where

$$\dot{q}_{\text{ang},1} = \left( \frac{m}{\sqrt{4I + mL}} \sqrt{m} \dot{y}(t_k^-) - \frac{2l}{L\sqrt{4I + mL}} \dot{\theta}(t_k^-) \right)$$

(6.71)

Recall that such singularities may pose some mathematical problems related to existence of solutions in certain evolution problems, see chapter 2, section 2.2, and chapter 5, section 5.3.
Hence it suffices that the angle $\theta_{1,2}$ (in the sense of the kinetic metric) between the two tangent vectors to the constraints at the singularity be larger than $\frac{\pi}{2}$ to render motion possible with this shock model (the tangential components remain continuous, the normal component has a discontinuity and its sign is changed, where everything is understood in the kinetic metric). This is illustrated in figure 6.7, where the Euclidean and the kinetic metrics are assumed to be equal for convenience.

Let us denote as $t_1 = \left( \begin{array}{c} 1 \\ -\frac{2}{L} \end{array} \right)$ and $t_2 = \left( \begin{array}{c} 1 \\ \frac{2}{L} \end{array} \right)$ the tangent vectors to both constraints in the Euclidean plane $(\theta, y)$, see figure 6.5. Then the preimpact velocity is along $t_2$, and motion is possible if and only if $(\dot{y}(t_2^-), \dot{\theta}(t_2^-)) \left( \begin{array}{cc} m & 0 \\ 0 & L \end{array} \right) t_1 \geq 0$. This yields after calculations the condition $4I \geq mL^2$, that is equivalent since $I = \frac{m}{12}(l^2 + L^2)$ to the condition

$$l^2 \geq 2L^2$$

(6.72)

Hence if (6.72) is verified, the model that consists of associating a restitution coefficient $e_n \in [0, 1]$ to the normal components of the velocity yields a feasible motion. It can be checked from (6.69) and (6.70) that this condition implies that after the shock, $\dot{y}(t_2^+) \geq 0$ whereas $\dot{\theta}(t_2^+) \leq 0$. Clearly such a motion is possible, since it means that the block continues to rotate in the same sense after the percussion, while the sign of $\dot{y}$ changes, and the left corner detaches from the surface. It can

Figure 6.6: Rocking motion.

Figure 6.7: Post impact motions at the singularity of the SRM.
also be verified that the condition in (6.72) can be retrieved by imposing \( \theta_{1,2} \geq \frac{\pi}{2} \). Indeed \( \theta_{1,2} = \arccos \left( \frac{L^2 m_2 - 4I}{\sqrt{(4 + L^2)(4I^2 + L^2 m_2)}} \right) \). (Such kinetic angle conditions are present in the sweeping process formulation, see [379] §5).

Before going on with these developments, let us calculate the equivalent interaction force acting on the block at the impact time, at the downwards edge middle point \( P = \left( \begin{array}{c} x_p \\ y_p \\ \theta \end{array} \right) \). The Jacobian between the vector of generalized coordinates derivatives \( \dot{p} = \left( \begin{array}{c} x_p \\ y_p \\ \theta \end{array} \right) \) and \( \dot{q} \) is \( J = \left( \begin{array}{ccc} 1 & 0 & \frac{1}{2} \cos(\theta) \\ 0 & 1 & \frac{1}{2} \sin(\theta) \\ 0 & 0 & 1 \end{array} \right) \). Hence the interaction wrench at \( P \) is given by

\[
p_p = J^{-T} p_q = \begin{pmatrix} 0 \\ \frac{2}{L} \\ -\frac{1}{L} \sin(\theta) + 1 \end{pmatrix} \lambda_1 \quad (6.73)
\]

where \( \lambda_1 \) has to be computed from (6.65) and (6.69) (6.70), and \( \lambda_1 > 0 \) for all \( e_1 \in [0, 1) \). Hence at \( P \) the interaction wrench is a vertical impulse \( \frac{2}{L} \lambda_1 \), together with an impulsive moment of magnitude \( \lambda_1 \) (recall that at impact \( \theta = 0 \)). Recall this has been obtained by assuming \textit{a priori} that \( p_q = \lambda_1 \nabla_q f_1(q) \); one sees that this assumption yields a physically sound result. At the lasting contact point, i.e. the left corner, it can be similarly calculated that the impulsion at the impact time is given by a vertical impulse \( \frac{2}{L} \lambda_1 \) and an impulsive moment \( 2\lambda_1 \). Notice that the impulsive moment is positive. This comes from the fact that the postimpact angular velocity is larger than the preimpact one, as one can check from (6.70). Another point is to investigate what is the real interaction between the block and the ground at the impact time.

\textbf{Remark 6.13} From (6.69) and (6.70), one sees that if the block rocks, i.e. rotates around \( B \) after the shock, we get \( f_1(q) = 0 \). Hence \( \dot{y}(t^+_k) = -\frac{L}{2} \dot{\theta}(t^+_k) \). Now since in this case \( \dot{y}(t^+_k) = \frac{L}{2} \dot{\theta}(t^+_k) \), we obtain from (6.69) and (6.70) that necessarily \( e_n = 0 \). Hence, given the assumption that \( \lambda_2 = 0 \), this restitution model describes rocking motion only with the most dissipative coefficient of restitution.

\textbf{Remark 6.14} Let us rewrite the block dynamics in coordinates \( z' = \left( \begin{array}{c} \theta \\ a' \\ b' \end{array} \right) \). We get

\[
\begin{align*}
\left( \frac{m}{2} (L^2 + I^2) + I \right) \sigma_{\theta}(t_k) + \frac{ml}{2} \sigma_{a'}(t_k) - \frac{ml}{2} \sigma_{b'}(t_k) &= 2\lambda_1 \\
\frac{ml}{2} \sigma_{\theta}(t_k) + m \sigma_{a'}(t_k) &= 0 \\
-\frac{ml}{2} \sigma_{\theta}(t_k) + m \sigma_{b'}(t_k) &= \frac{2}{L} \lambda_1 + \lambda_2
\end{align*}
\quad (6.74)
\]
(Note that this set of dynamical equations is equivalent to (4.25), where one of the bodies is fixed, with infinite mass. Moreover the matrix $M_i$ in (4.11) is given simply here by the Jacobian between $\dot{q}$ and $\dot{z}$ when $i = B$). From (6.74) one sees that evidently, one restitution coefficient is not sufficient to calculate all the unknowns, except if one makes extra-assumptions as we did in section 6.4.1. Which coefficients have to be chosen then? In subsection 6.5.2 the choice of two generalized coefficients is made. In fact only experiments can validate the choice of a restitution rule. See also remark 6.22.

6.5 General restitution rules for multiple impacts

In this section, we continue to analyze the possibility of defining restitution rules to cope with the multiple impacts case. Contrarily to subsection 6.4.1, we search for a solution that encompasses all possible preimpact configurations.

6.5.1 Introduction

Let us consider again the rolling block example analyzed in section 6.4.1. Now what about the case when the model is not valid, because $\theta_{1,2} < \frac{\pi}{2}$? In other words, the proposed model is not able to provide a feasible postimpact motion, and one has to search for something else. At this point, two main ideas come to mind: when striking at a singularity, the normal direction is multiply defined (By the different hypersurfaces that compose the singularity). Then it is possible to define a matrix $n_q$ of normal vectors, and write down the transformed dynamics in (6.6). The idea is then to associate a restitution coefficient to each one of these normal directions, and analyze the postimpact motions. But notice at once that if the preimpact motion is along one of the hypersurfaces (or grazes it), then the corresponding normal preimpact velocity must be zero, and applying a restitution Newton's like rule makes no sense. This idea can be retained when the singularity is reached without any grazing motion. But it may involve further problems, see subsection 6.5.8. One of them is the following, that can be clearly explained: assume that the constraint is of codimension 2. Then two vectors $n_{q,1}$ and $n_{q,2}$ can be defined. The corresponding dynamical equations at the shock instant will be (see (6.6))

$$
\begin{align*}
\dot{q}_{\text{norm},1}(t_k^+) - \dot{q}_{\text{norm},1}(t_k^-) &= n_{q,1}^T p_q \\
\dot{q}_{\text{norm},2}(t_k^+) - \dot{q}_{\text{norm},2}(t_k^-) &= n_{q,2}^T p_q \\
\dot{t}_{\text{tang}}(t_k^+) - \dot{t}_{\text{tang}}(t_k^-) &= 0
\end{align*}
$$

\(^{8}\text{Note that striking at a singularity is exactly the same as striking a constraint of codimension} \geq 2.\)}}
where \( p_q = \lambda_1 \nabla_q f_1(q) + \lambda_2 \nabla_q f_2(q) \), \( \lambda_1 \in \mathbb{R}^+, \lambda_2 \in \mathbb{R}^+ \). Now assume that one associates a Newton restitution rule to each normal component, i.e. 

\[
\dot{q}_{\text{norm},1}(t_k^+) = -\epsilon_1 q_{\text{norm},1}(t_k^-) \quad (6.76)
\]

and

\[
\dot{q}_{\text{norm},2}(t_k^+) = -\epsilon_2 q_{\text{norm},2}(t_k^-) \quad (6.77)
\]

Assume that \( q_{\text{norm},1}(t_k^-) = 0 \), which means that the preimpact trajectory grazes the surface \( f_1(q) = 0 \). Normally, according to (6.76), one should get \( \dot{q}_{\text{norm},1}(t_k^+) = 0 \). But now a look at (6.75) shows that in general, one gets \( \dot{q}_{\text{norm},1}(t_k^+) = n_{q,1}^T p_q \) which has no reason to be zero, because in general \( n_{q,1}^T \nabla_q f_2(q) \neq 0 \). Hence a contradiction which shows that defining such restitution rules for multiple impacts is in general incoherent. Orthogonality of the constraint (with respect to the kinetic metric) surfaces guarantees coherence (see subsection 6.5.8). However, let us disregard some wellposedness of the dynamical equations (existence, uniqueness, continuity with respect to initial data) problems. It is possible to use a restitution rule as in (6.76) and (6.77), if one accepts that the coefficients are not universal, but depend on the preimpact velocities for instance. In fact, there are real physical systems for which multiple impacts occur (see the 3-balls, the rocking block examples), and with non-orthogonal constraints. For those systems, we need a model to analyze and predict postimpact motion.

In this section, we shall just try to answer to the question: can we describe exhaustively the possible postimpact outcomes with a set of constant\(^9\) restitution coefficients? Let us consider the problem from another point of view than in (6.75). Let us choose one of the hypersurfaces that compose the singularity, and let us assume that at the shock instant, the tangential velocities can reverse. In other words, we again follow the philosophy described in chapter 4. From equations (4.25) and (4.44) we noticed that \( p_q \) and the restitution coefficients to be defined are related. Note that when the system strikes at a singularity, then we no longer obtain \( \sigma_{\dot{q}_{\text{tang}}}(t_k) = 0 \), because \( t_i^T q_i p_q \neq 0 \) for some \( i \), since \( p_q \) is no longer along \( \nabla_q f_1(q) \). Hence we have to define a restitution coefficient for this component of the tangential velocity. We warn the reader that here we somewhat leave scientific rigor to penetrate into the realm of engineering tricks that allow to associate a model to an experimental evidence (In some experiments the block rocks until it stabilizes at the singularity). Recall however that the association of restitution coefficients to complete the dynamical equations is in a certain sense the only way to proceed with. One really needs to relate post and preimpact velocities to solve the dynamical problem. In the rocking-rolling block example, the singularity may be attained with a motion sliding on \( f_2(q) = 0 \), and with constraints which are not at all orthogonal. Even in this case, it would be nice to define a restitution rule. Indeed such conditions do not call into question the fact that the system is rigid or not. Let us choose the

\(^9\)It has to be precised what is meant by constant. As we shall see below, it is not always evident that one is able to fix the value of the coefficients (independently of the preimpact velocities), and at the same time describe a certain motion with no energy loss.
constraint $f_i(q) = 0$ to define the normal and tangential directions. By doing so we have enough degrees of freedom to find coefficients such that the whole configuration space is spanned. From (6.6), one sees that it is possible to associate restitution coefficients to both $q_{\text{norm}}$ and $q_{\text{tang}}$, i.e. in fact an overall maximum number of $n$ coefficients. If they are denoted as $e_n$ and $e_{t,i}, 1 \leq i \leq n - 1$ (recall we deal for the moment with codimension one constraints), then we have (provided the tangential and normal vectors $n_q$ and $t_{q,i}$ are chosen unitary and mutually orthogonal for the kinetic metric)

$$T_L = \frac{1}{2}(e_n^2 - 1) \left( q_{\text{norm}}(t_k^-) \right)^2 + \frac{1}{2} \sum_{i=1}^{n-1} (e_{t,i}^2 - 1) \left( q_{\text{tang}}(t_k^-) \right)^2$$

(6.78)

If one assumes that the set of restitution coefficients must guarantee $T_L \leq 0$ for any initial data, then it is clear that the coefficients must belong to $[-1, 1]$. It might also be argued that they can be outside this interval for particular processes. Are they to be considered as independent of the initial conditions while satisfying the energy criterion, or should they depend on preimpact velocities? The only thing that we can say now is that the energy constraint, the form of the percussion vector ($p_q \in \text{span}(\nabla_q f_i(q))$) and the postimpact motion constraint imply that the coefficients in (6.78) belong to some compact subset of $\mathbb{R}^n$.

Thus viewing the system as a point in its configuration space allows in theory to thoroughly investigate postimpact possible motions. A postimpact velocity yields a possible motion if and only if it belongs to (or points inwards) the domain of possible motion. Given any Lagrangian system, it is possible to describe the domain of motion and its boundary. Hence it is possible to check whether the calculated velocities have a right direction or not. Obviously, the more complex the system, the less easy the computations. For a system composed of numerous bodies like the ones considered in [187], the complete dynamical equations and constraints may rapidly become complex and render the method less tractable numerically. But this is similar for any method, unless simplifying assumptions are done. The main problem is to determine the possible values of the coefficients. It is not possible to test all the values that yield a feasible motion, because there are in general an infinity of such values. For instance, if the system strikes the domain’s boundary at a smooth point, then it is sufficient to impose a normal rebound, and (6.13) imposes a value for the coefficient. Indeed at a smooth point of the boundary the normal direction is uniquely defined. Hence locally around such a point the constraint is of codimension one, $q_{\text{norm}}$ is scalar and there is one coefficient only. If the shock occurs at a singularity, several cases have to be considered. The system may slide on one of the constraint and strike another one: then it seems logical to consider the impact between the system and the attained constraint, from which the normal and tangential directions can be defined. If the system attains the singularity directly (i.e. without touching the domain’s boundary), then one may choose a rebound

\[10\] In the sense that even if it is always possible to define tangential restitution, one may impose from experimental considerations the form of the total reaction, to a priori preclude tangential rebound.
6.5. GENERAL RESTITUTION RULES FOR MULTIPLE IMPACTS

along the bissector of the angle (in dimension 2), or along one of the constraints that form the singularity. For instance it is apparent that for the rolling block, the only way to attain the singularity without striking one of the constraint surfaces is to move along the y-axis with \( \theta = 0 \). Then one may logically choose the postimpact motion along that axis, which means that the block strikes the constraint through its whole downwards edge and rebounds horizontally. At this stage we are not able to go further in the determination of ranges of values for the restitution coefficients. Note that the rules to be chosen may crucially depend on the initial conditions, i.e. preimpact velocity. This is not surprising and is a generalization of the fact that in a one-dimensional problem, the restitution coefficient is likely to depend on initial conditions also. The only thing we can say is that it is not possible to solve the impact problem without stating some \textit{a priori} rules. These rules may come from experimental evidence. They must yield a coherent model, in particular \( T_L \leq 0 \). The approaches in [187] [222] possess the advantage of proposing a framework that enables to define possible postimpact directions. This is also the case for the sweeping process formulation. It has however to be checked that these models fit with reality. In particular simulation results in [222] show that different outcomes occur depending on the models of impact available in the literature. Certainly they cannot be all good at the same time!

In the examples we analyze below, we shall see the problems in more detail. Note that the case \( T_L = T_{L,\text{max}} \) corresponds to generalized dissipative shocks, but is different in general from the sweeping process: the sweeping process imposes that the postimpact velocity be closest to the preimpact one and yields a feasible motion, whereas taking all the coefficients zero here simply means that the system is at rest after the shock. Following remark 4.15, we can also impose a lowerbound for \( T_L \) to allow possible postimpact velocity.

6.5.2 The rocking block example continued

Let us now come back to the rocking block example in section 6.4.1. Namely, to render motion possible, we may impose a tangential restitution coefficient to \( \dot{q}_{\text{ang},1} \), i.e. (recall that this is a 2-dimensional vector, so we may in general define two such coefficients)

\[
\dot{x}(t_k^+) = -e_{t,z}\dot{x}(t_k^-) \tag{6.79}
\]

\[
m\ddot{y}(t_k^+) - \frac{2I}{L}\dot{\theta}(t_k^+) = -e_t \left( m\ddot{y}(t_k^-) - \frac{2I}{L}\dot{\theta}(t_k^-) \right)
\]

We allow for negative tangential coefficients to encompass the case when no tangential velocity reversal is needed, like when \( \theta_{1,2} \geq \frac{\pi}{2} \). From (6.64) (6.79) we obtain

\[
\dot{\theta}(t_k^+) = \frac{2mL(e_1 - e_2)\dot{y}(t_k^-)}{4I + mL^2} - \frac{mLe_4 + 4le_5}{4I + mL^2}\dot{\theta}(t_k^-)
\]

\[
= \frac{e_1L^2m - 2e_2L^2m - 4le_5}{4I + mL^2}\dot{\theta}(t_k^-)
\]
and
\[ \dot{y}(t_k^+) = -\frac{4I e_n + mL^2 e_t}{4I + mL^2} \dot{y}(t_k^-) + \frac{2II(e_t - e_n)}{4I + mL^2} \dot{\theta}(t_k^-) \]
\[ = \frac{-mL^2 e_t + 4I(e_t - 2e_n)}{4I + mL^2} \dot{y}(t_k^-) \]  
\[ (6.81) \]
It can be checked that (6.80) and (6.81) reduce to (6.68) and (6.67) respectively for \( e_t = -1 \), which corresponds to zero jump in the second component of \( \dot{q}_{\text{tang},1} \). Rocking motion is still described by \( e_n = 0 \) (the normal component \( \dot{q}_{\text{nor}}(t_k^+) \) must be zero for the system to evolve on \( f_1(q) = 0 \) after the shock), in which case (6.80) and (6.81) become
\[ \dot{\theta}(t_k^-) = \frac{mL^2 - 4I}{4I + mL^2} e_t \dot{\theta}(t_k^-) \]  
\[ (6.82) \]
and
\[ \dot{y}(t_k^-) = \frac{4I - mL^2}{4I + mL^2} e_t \dot{y}(t_k^-) \]  
\[ (6.83) \]
It is clear from (6.82) and (6.83) (see also (6.78)) that \( e_t \) may be used to fix the kinetic energy loss \( T_L \). However notice that if one takes \( e_t \approx 1, \) then necessarily \( T_L < 0 \). In order to be able to encompass rocking with no energy loss, one must choose \( e_t > 1 \). And this value depends on \( \dot{q}_{\text{nor}}(t_k^-) \). Indeed a quick look at (6.78) shows that \( T_L = 0 \) if
\[ e_t = \frac{\sqrt{\left( \dot{q}_{\text{nor}}(t_k^-) \right)^2 + \left( \dot{q}_{\text{tang},12}(t_k^-) \right)^2}}{|\dot{q}_{\text{tang},12}(t_k^-)|} \]  
\[ (6.84) \]
where \( \dot{q}_{\text{nor}}(t_k^-) = \frac{\sqrt{mL}}{\sqrt{4I + mL^2}} \left( \frac{2}{L} \dot{y}(t_k^-) + \dot{\theta}(t_k^-) \right) \) and \( \dot{q}_{\text{tang},12}(t_k^-) = \frac{\dot{y}(t_k^-) - 2I \dot{\theta}(t_k^-)}{\sqrt{4I + mL^2}} \).

A question that arises then is the following: is it possible to describe rocking motion, with various kinetic energy losses, and with a unique set of constant coefficients \( e_t \)? Constant means here that if one fixes for instance \( T_L = 0 \), can this be obtained for any preimpact velocity value, with a unique coefficient \( e_t \) as it is the case for a codimension one constraint? From (6.78) it is apparent that \( e_n = e_t = 1 \) are the only constant values of the coefficients that yield \( T_L = 0 \) independently of the preimpact velocity. But this does not describe rocking, because \( e_n = 1 \) means that \( \dot{b}'(t_k^+) > 0 \) after a shock at corner \( B \).

Recall that the a priori introduction of additional coefficients imposes a percussion vector that acts on the system at the shock instant. The dynamical equations in (6.6) when the tangential coefficients are introduced become
\[ \begin{cases} 
-(1 + e_n) \left( \frac{2}{L} \dot{y}(t_k^-) + \dot{\theta}(t_k^-) \right) = p_1 \\
-m(e_x + 1) \dot{x}(t_k^-) = p_2 \\
-(e_t + 1)(m \dot{y}(t_k^-) - 2I \dot{\theta}(t_k^-)) = p_3 
\end{cases} \]  
\[ (6.85) \]
It clearly appears from (6.85) that the total percussion vector at the impact time is modified. However the fact that \( p_q = \lambda_1 \nabla_q f_1(q) + \lambda_2 \nabla_q f_2(q) \) for \( \lambda_1, \lambda_2 \in \mathbb{R}^+ \),
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it can be calculated that \( \left( \frac{n_q^T}{t_q} \right) p_q \) yields necessarily \( p_q = 0 \), i.e. \( \dot{x} \) is continuous. It is possible now to compute the set of coefficients \( e_n \) and \( e_t \) that yield a feasible postimpact motion, i.e. \( T_L \leq 0 \) and \( \dot{q}(t_k^+) \) points inwards \( \Phi \). Recall that for a codimension one constraint, the second condition is simply \( \dot{q}^T(t_k^+) \nabla_q f(q) \geq 0 \leftrightarrow \dot{q}_{\text{norm}}(t_k^+) \geq 0 \). When a singularity is attained, we saw in chapter 5, section 5.3, that the postimpact velocity must lie in the cone \( V(q) \) (see definition 5.1 and proposition 5.1). We can translate this into \( \dot{q}_{\text{norm},1} \geq 0 \) and \( \dot{q}_{\text{norm},2} \geq 0 \), where both components are calculated with the respective normal vectors \( n_{q,1} \) and \( n_{q,2} \) to each constraint.

Remark 6.15 Is there a change of generalized coordinates \( X = X(q) \), that allows to avoid the singularity? At a singularity we have \( \nabla_q f_1(q) \neq \nabla_q f_2(q) \). Let us denote \( X = J(q)\dot{q} \). Then \( f_1(q) = f_0 X^{-1}(X) = g_1(X) = g_0 X(q) \), and \( J^T(q) \nabla_X g_1(X) = \nabla_q f_1(q) \) since the gradient transforms covariantly. Hence we get \( \nabla_X g_1(X) \neq \nabla_X g_2(X) \). Singularities are invariant with respect to the generalized coordinates.

Remark 6.16 This example proves that it is not possible to choose constant coefficients to describe a particular motion (rocking with \( T_L = 0 \)) for any initial data. This justifies the use of other formulations without restitution coefficients like complementarity formulations, see chapter 5, section 5.4.

6.5.3 Additional comments and studies

Housner [212] [213] has apparently been the first to derive the mathematical equations of a free-standing rigid block under base excitation. No restitution coefficient is used in [213], who rather quantifies the kinetic energy loss at impact. The rocking block problem has also received attention in [54] §6.6, who discusses the equivalent wrench value of the distributed forces acting on the block surface. An approximating compliant problem with distributed spring-like and damper-like actions is also presented, and the possibility of defining a moment restitution coefficient is discussed. Moreau [385] proposes in the framework of a complementarity dynamical formulation, a trick that allows to solve that problem. We described it in section 5.4. Aslam et al [21] simply define an angular restitution coefficient as the ratio \( e_a = \frac{\dot{\theta}(t_k^+)}{\dot{\theta}(t_k^+)} \) and determine experimentally this value for two different concrete blocks, with \( L = 0.152 \) m and \( l = 0.762 \) m, \( L = 0.229 \) m and \( l = 0.974 \) m. They find that \( e_a = 0.925 \) for each block, by performing free rocking tests. The values computed from (6.70) provide a restitution coefficient of 0.794 and 0.728 respectively (with \( e_n = 0 \), since the model described in subsection 6.4.1 describes rocking for this value only). This might show that the assumptions made in subsection 6.4.1 are not satisfied in those experiments, although condition (6.72) is satisfied. But recall that the model chosen in subsection 6.4.1 are very simple, i.e. no friction and no slipping of the base on the ground. From (6.70), it is legitimate to use a restitution coefficient as \( e_a \), although this is not justified by these authors. Recall
that the definition of only one restitution coefficient implies some priori knowledge on the percussion vector, see subsection 6.4.1. However the value for $e_a$ found in [21] with the two blocks described above provide two different values of $e_n$ in (6.70). It might be that a tangential restitution has to be introduced and that the unique coefficient used in [21] is in fact a combination of both $e_t$ and $e_n$ as in (6.80).

Lipscombe et al [314] discuss the validity of defining the angular restitution from conservation of momentum at the right corner of the block like in [213], which provides $e_n = \frac{2L^2 - L^2}{2L^2 + 2L^2}$. Theoretical values calculated from this method and experimental ones are compared, and are not found to fit very well in general (see table 1 in [314], where the experimental results found in [21] [390] and [445] are compared to those predicted via $e_a$ computed as above). Motivated by these results, the authors [314] consider another way to calculate the restitution coefficient, introducing a second coefficient defined as $e = \frac{2\theta^*}{2\theta^* - \omega_x}$. Note that this is simply the ratio $\frac{\theta(t_k)}{\theta(t_k^*)}$, where $\theta$ is the vertical coordinate of point B in figure 6.3. Experimental results [315] on steel blocks colliding with a steel base show that $e = 0.9$, when $\theta = 0$ and for low velocities. Applying this rule one exactly recovers (6.64), i.e. $e = e_n$ in (6.64). Then they calculate $e_a = \frac{2L^2 - L^2 - 3L^2}{2L^2 + 2L^2}$, retrieving a result in [490].

This expression is obtained assuming that $x$ is discontinuous at $t_k$ (hence necessarily friction is considered, since we have seen that with frictionless constraints then $x$ remains continuous, from the form of the percussion vector), and using the kinematic relationships from rocking conditions. The main conclusion in [314] is that the rigid model of the rocking block together with the restitution coefficients as defined in [314], provides good prediction for a certain range of values of $\frac{L}{h}$ only. This is supported by a very detailed analysis of the shock process, that may hide very subtle phenomena (like slipping of the base). Frémond [160] assumes that the percussion is given by

$$P = P(v(t_k^*), v(t_k)) = -k(v_n(t_k^*) + v_n(t_k)) + R^+ \text{ if } v_n(t_k^*) = 0,$$

$$\emptyset \text{ if } v_n(t_k^*) < 0,$$

where $v_n(\cdot)$ is the normal velocity of the contact point, and $k \geq 0$. Then various types of motion are described. For instance, rocking occurs for certain values of the parameters $m, L, l$ and $I$, and with $e_a = \frac{4L^2 - mL^2}{4L^2 + 4KL^2 + mL^2}$. An important feature of the percussion law proposed in [160] is that it is energetically consistent for all $k \geq 0$. Other developments can be found in [498].

The motion of the rocking block subject to base excitation is studied in various papers [242] [205] [203] [204] [79] [507] [589] [202] [599] [600] [285] [601] [545]. The SRM is used, with an angular restitution. It is shown that very complex dynamics like chaotic response can be obtained. These studies are important to
predict the possible motions of various types of buildings during earthquakes.

**Remark 6.17** Notice that, contrarily to what intuition might tell, the fact that \( e_n = 0 \) alone does not mean at all that the "real-world" percussion of the block (i.e. what is observed in reality) is purely dissipative, i.e. motion stops after the shock: we have seen that it corresponds to the block starting to rock around \( B \). Such phenomenon requires more than that in general (From (6.67) (6.68) it occurs if \( mL^2 = 4I \), i.e. \( I^2 = 2L^2 \)). As we saw, it requires that both tangential and normal restitution be zero. Note also that if the block is not square, then the global motion may involve not two, but four constraints on the orientation, depending on which face is going to strike the plane (see figure 6.8).

**Remark 6.18** Instead of defining a restitution coefficient for each component of the transformed generalized momentum in (6.5), we could have also defined a restitution operator \( E \) such that:

\[
\begin{bmatrix}
\dot{q}_{\text{norm}}(t_k^+)&
\dot{q}_{\text{tang}}(t_k^+)
\end{bmatrix}
E
\begin{bmatrix}
q_{\text{norm}}(t_k^-)
q_{\text{tang}}(t_k^-)
\end{bmatrix}
\]

(6.86)

with the requirement that \( T_L \leq 0 \) (see section 7.1.3 for more details on the passivity properties of the impact map). Recall that the form of \( E \) also depends on the percussion vector.

### 6.5.4 3-balls example continued

Let us now re-examine in detail the 2 and 3 balls examples that we described in chapter 5. From now on we have led calculations for the lamina and the rocking block example. They aimed at proving that various motions can be represented via some restitution coefficients, acting on generalized velocities in the configuration space (recall that we do not at all care about mathematical existential or continuity with respect to initial data in this chapter). We now generalize these ideas on the classical balls example, which represents a system with multiple impacts. It also possesses the advantage that everyone can check if theory fits with experiments since such devices are not difficult to build (or buy!). In section 5.2 we have seen that some
rigid body algorithms yield in general several outcomes for one set of preimpact velocities. We have also seen for the 3 balls case that there are two solutions, and that they correspond to two limit cases, when one places some flexibility in the system (see section 5.2.1). We generalize these ideas now.

For the 3-balls case, the unilateral constraints in the \((q_1 - q_2), (q_1 - q_3)\) and \((q_3 - q_2)\)-planes are depicted in figure 6.9. They are given by:

- \(f_1(q) = q_1 - 2R = q_2 - q_1 - 2R \geq 0\),
- \(f_2(q) = q_2 - 2R = q_3 - q_2 - 2R \geq 0\),
- \(f_3(q) = q_3 - 4R = q_3 - q_1 - 4R \geq 0\).

They are not independent since \(\frac{\partial f_3}{\partial q_1} = \frac{\partial f_2}{\partial q_1} - \frac{\partial f_1}{\partial q_1}\). On each figure we have also depicted the preimpact velocity and the postimpact ones as calculated in (5.6) and (5.8). Since bodies 2 and 3 are in contact before the impact, the preimpact velocity is tangential to \(f_2(q) = 0\).

Note that in an \(n\)-balls system, the boundary of the domain \(\Phi\) is given by the union of \((n - 1)\) hyperplanes \(\partial \Phi_1 \cup \cdots \cup \partial \Phi_{n-1}\), with \(\partial \Phi_i = \{q : q_i = q_{i+1}\}\). The smooth or regular part \(\partial \Phi_i^s\) of \(\partial \Phi_i\) is given as \(\partial \Phi_i^s = \{q : q \in \partial \Phi_i, \tilde{q}_k \neq 0, \tilde{q}_k = q_k - q_{k-1}, k \neq i\}\). The singularities of \(\partial \Phi\) are codimension \(\geq 2\) subspaces of \(\partial \Phi \subset \mathbb{R}^n\), given by \(\partial \Phi_j^s = \{q : \tilde{q}_i = \cdots = \tilde{q}_k = 0\text{ for some } 1 \leq i, \cdots, k \leq n - 1\}\).

In the following, we first define an arbitrary restitution rule that contains as a
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particular case one of the solutions found by applying the algorithm in [187] (see chapter 5). Then we describe a way to obtain a set of all admissible generalized coefficients, which define the restitution operator relating generalized post to preimpact velocities. Finally we investigate the dynamics of an approximating problem, which justifies the preceding developments (Recall that we have already seen that two particular approximating problems converge to the solutions found by the algorithm in [187], see section 5.2.1. This path is generalized).

A possible restitution law

Due to the symmetry of the problem, we are tempted to choose the postimpact velocity along the bissector of \( f_1(q) = 0 \) and \( f_3(q) = 0 \). Also we choose the vector of coordinates as \( q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \). We can interpret this choice as follows: since the normal vector to the constraint at the singular point is not uniquely defined, we choose to define it as the bissector to the normal vectors to \( f_1(q) = 0 \) and \( f_3(q) = 0 \). Then we apply a restitution rule along this new "average" normal direction.

This direction is given by \( n = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \). \( q \) could have been chosen else, but this choice yields simple further calculations. Tangential directions can be chosen as \( t_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \) and \( t_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \). Then it follows that \( \dot{q}_{\text{norm}} = \dot{q}_2 - 2\dot{q}_1 + \dot{q}_3 \), and  

\[
\dot{q}_{\text{tang}} = \begin{pmatrix} -x_2 + x_3 \\ -x_2 + x_1 - x_3 \\ 0 \end{pmatrix}.
\]

Let us apply the rule \( \dot{q}_{\text{norm}}(t_k^+) = e_1 \dot{q}_{\text{norm}}(t_k^-) \), \( \dot{q}_{\text{tang}}(t_k^+) = e_2 \dot{q}_{\text{tang}}(t_k^-) \). We choose a zero restitution for the first tangential component because its value is immaterial for the particular set of initial conditions (This tangential component is initially zero). We obtain after calculations since the inertia matrix in \( q \)-coordinates is the identity (recall the masses are equal to one)

\[
\dot{q}_1(t_k^+) = \frac{2e_2 + e_1}{2} \quad \dot{q}_2(t_k^+) = \dot{q}_3(t_k^+) = \frac{3}{2}(e_2 - e_1)
\]

(6.87)

From \( T_L = 0 \) it follows that the coefficients are constrained by

\[
\left( e_2 - \frac{e_1}{2} \right)^2 + \frac{e_1^2}{4} = \frac{3}{4}
\]

(6.88)  

and by the fact that postimpact motion must be feasible, i.e. \( \dot{q}_2(t_k^+) = \dot{q}_3(t_k^+) \geq \dot{q}_1(t_k^+) \). This implies \( e_1 \leq 0 \). This is still not sufficient to guarantee uniqueness of the solution. This can be obtained if one imposes the form of the interaction wrench impulse. The solution in (5.6) corresponds to \( e_2 = 0 \) and \( e_1 = -1 \).

This is therefore a generalized elastic rebound along \( n \). It is note worthy that the projection of the preimpact velocity on \( t_1 \) is zero. Therefore it is not possible
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to describe all eventual postimpact motions with this choice of projections and associated coefficients.

General algorithm

We have chosen more or less arbitrarily a generalized collision rule for a multiple impact between three balls. It is possible to generalize these ideas. Indeed, our primary goal in this section is to derive a method that allows to describe all the possible postimpact motions with a set of coefficients. In fact, for the case of a simple impact, the choice of the basis \( n_q, t_q \) is a natural one, as it directly extends to \( n \)-degree-of-freedom systems the case of two bodies colliding. But for multiple impacts, there is no such natural extension, as the above developments on the rocking-block problem and the striking-balls have shown. Hence one can a priori use another basis, provided it allows to fulfill the above requirements. The vector of generalized coordinates is chosen as \( \vec{q} = (\vec{q}_1, \vec{q}_2, \vec{q}_3)^T \). The corresponding inertia matrix is given by \( \vec{M} = M(\vec{q}) = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 3 \end{pmatrix} \), and the percussion vector by \( (-p_{21}, -p_{32}, 0)^T \), where \( p_{ij} \) is the impulsive action of \( i \) on \( j \). Let us propose the following algorithm:

1. Determine 3 vectors \( n_1, n_2 \) and \( n_3 \in \mathbb{R}^3 \) such that \( \vec{q}(t^*_k)^T \vec{M} n_i \neq 0, i = 1, 2, 3, \) and such that \( (n_1, n_2, n_3) \) is \( \vec{M} \)-orthonormal. Let us denote \( E = \begin{pmatrix} n_1^T \\ n_2^T \\ n_3^T \end{pmatrix} \in \mathbb{R}^{3 \times 3} \), and \( \vec{q}^* = EM \vec{q} \). To each component \( \vec{q}_i^* \) associate a coefficient \( e_i \), i.e. \( \vec{q}_i^*(t^*_k) = e_i \vec{q}_i^*(t^-_k) \).

2. There is a kinetic energy loss, i.e. \( T_L \leq 0 \). This implies that the \( e_i \)'s lie in a subspace \( S_T \) of \( \mathbb{R}^3 \).

3. The postimpact motion must be feasible, i.e. \( \dot{\vec{q}}(t^*_k) \) points inwards the domain \( \Phi \), i.e. \( \dot{\vec{q}}(t^*_k)^T \vec{\nabla} f_i(\vec{q}) \geq 0, i = 1, 2 \). This implies that the \( e_i \)'s lie in a subspace \( S_v \) of \( \mathbb{R}^3 \).

4. The \( e_i \)'s must be chosen in accordance with the form of the percussion vector \( P_q \). This implies that the \( e_i \)'s lie in a subspace \( S_p \) of \( \mathbb{R}^3 \).

5. The admissible coefficients \( e_i \in \mathbb{R} \cap \mathbb{R} \cap \mathbb{R} \).

Condition i guarantees that the whole space \( \mathbb{R}^3 \ni \dot{\vec{q}}(t^*_k) \) can be spanned for any preimpact velocity, by taking varying \( e_i \)'s. Also the kinetic energy is given by \( T = \frac{1}{2} \dot{\vec{q}}^T \dot{\vec{q}} \). Note that \( \sigma_q^+(t_k) = E \sigma_q^+(t_k) = EM P_q \), where \( P_q \) is the generalized percussion vector in coordinates \( \vec{q} \).

Let us apply the algorithm step by step.
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i) The preimpact velocity is given by 
\[ \dot{\mathbf{q}}(t^-) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]. It can be calculated that 
\[ \dot{\mathbf{q}}(t^-)^T \tilde{M} \mathbf{n}_{q,1} = \frac{1}{\sqrt{2}} \], while 
\[ \dot{\mathbf{q}}(t^-)^T \tilde{M} \mathbf{n}_{q,2} = 0 \], where \( \mathbf{n}_{q,i} \) is computed following (6.2), for the surface \( f_i(q) = 0 \). Consequently another set of vectors has to be defined. It can be checked that the basis \( b_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, b_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{3} \\ -\sqrt{3} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}, b_3 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{3}} \end{bmatrix} \) is \( \tilde{M} \)-orthonormal. Also one computes that
\[ \dot{\mathbf{q}}^* = E\tilde{M} = \begin{bmatrix} \frac{\dot{q}_1}{\sqrt{2}} + \frac{\dot{q}_2}{\sqrt{2}} \\ \frac{\dot{q}_3}{\sqrt{6}} - \frac{\dot{q}_2}{\sqrt{6}} \\ -\frac{\dot{q}_1}{\sqrt{3}} - 2\frac{\dot{q}_2}{\sqrt{3}} + \sqrt{3}\dot{q}_3 \end{bmatrix} \] (6.89)
The corresponding preimpact value is 
\[ \dot{\mathbf{q}}^*(t^-) = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \] and the postimpact value is given by
\[ \dot{\mathbf{q}}^*(t^+) = \begin{bmatrix} \frac{\dot{q}_1}{\sqrt{2}} \\ \frac{\dot{q}_3}{\sqrt{6}} \\ -\frac{\dot{q}_2}{\sqrt{3}} \end{bmatrix} \] (6.90)

ii) Let us impose no kinetic energy loss at impact. This gives the set \( S_T \):
\[ 3e_1^2 + e_2^2 + 2e_3^2 = 6 \] (6.91)

iii) Feasibility of the postimpact motion \( (\dot{\mathbf{q}}(t^+)^T \nabla_q f_i(q) \geq 0, i = 1, 2) \) implies
\[ e_1 + e_2 \geq 0 \quad e_1 - e_2 \geq 0 \] (6.92)
which define the set \( S_v \).

iv) The percussion vector is given by
\[ \mathbf{P}_q = E^{-1}\sigma q(t_k) = \begin{bmatrix} -\frac{1+e_1}{\sqrt{2}} \\ \frac{1+e_2}{\sqrt{6}} \\ -\frac{1+e_3}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} -p_{21} \\ -p_{32} \\ 0 \end{bmatrix} \] (6.93)
From the fact that \(-p_{21} \geq 0\) and \(-p_{32} \geq 0\) one obtains the set \( S_p \):
\[ \frac{1+e_1}{2} + \frac{1+e_2}{6} \geq 0 \quad \frac{1+e_1}{2} - \frac{1+e_2}{6} \geq 0 \quad e_3 = -1 \] (6.94)
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The set of admissible generalized restitution coefficients \( S_T \cap S_c \cap S_p \) is depicted in figure 6.10. It is noteworthy that the two solutions found by the algorithm in [187] (see (5.6) and (5.7)) correspond to the two extreme points \((e_1, e_2) = (-1, 1)\) and \((1, 1)\). They also correspond to two limit approximating problems (see section 5.2.1).

**Approximating problem**

It is clear that one needs to provide some physical significance to the coefficients introduced above. As we saw in chapter 2, an elegant manner to justify restitution laws is to consider the rigid problem as the limit of a sequence of compliant problems. In section 5.2.1 we have studied two particular approximating problems, and the general model has been presented in (5.17). This paragraph is devoted to study the limit of the system in (5.17) when both stiffnesses tend to infinity.

Let us take the initial time when balls make contact as \( t = 0 \). Then on an interval \([0, \delta]\), \( \delta > 0 \), both springs are crushed and the dynamical equations are given by:

\[
\begin{align*}
\ddot{q}_1 &= k(q_2 - q_1) \\
\ddot{q}_2 &= k(q_1 - q_2) + \alpha k(q_3 - q_2) \\
\ddot{q}_3 &= \alpha k(q_2 - q_3)
\end{align*}
\]

(6.95)

It can be seen that on \([0, \delta]\) one has

\[
\dot{q}_1 + \dot{q}_2 + \dot{q}_3 = 0
\]

(6.96)

Let us denote \( z = q_1 + q_2 + q_3 \). Thus we have on \([0, \delta]\): \( \ddot{z} = 0 \), with \( z(0) = 0 \), \( z'(0) = 1 \). Hence

\[
z(t) = t
\]

(6.97)

This relationship is valid until two of the balls separate. For the moment let us assume that we are on \([0, \delta]\), and let us calculate the expressions of the balls’ positions. From (6.95) and (6.97) one finds:

\[
\begin{align*}
\dot{q}_2 &= k(t - 2q_2 - q_3) + \alpha k(q_3 - q_2) \\
\dot{q}_3 &= \alpha k(q_2 - q_3)
\end{align*}
\]

(6.98)
which we rewrite as

\[
\begin{cases}
\dot{q}_2 = k t - k(2 + \alpha)q_2 + k(\alpha - 1)q_3 \\
\dot{q}_3 = \alpha k q_2 - \alpha k q_3
\end{cases}
\]  

(6.99)

Let us now take the following notations:

\[
\begin{align*}
\alpha_1 &= k(\alpha + 1 + \sqrt{\alpha^2 - \alpha + 1}) \\
\alpha_2 &= k(\alpha + 1 - \sqrt{\alpha^2 - \alpha + 1}) \\
\alpha_3 &= \alpha + 1 + \sqrt{\alpha^2 - \alpha + 1} \\
\alpha_4 &= \alpha + 1 - \sqrt{\alpha^2 - \alpha + 1} \\
\alpha_5 &= \sqrt{\alpha^2 - \alpha + 1}
\end{align*}
\]  

(6.100)

Notice that \(\alpha^2 - \alpha + 1 > 0\) for all \(\alpha \in \mathbb{R}^+\). Then it follows that:

\[
\begin{align*}
q_1(t) &= \frac{t}{3} + \frac{2\alpha - \alpha_3}{2\alpha_3 \alpha_5 \sqrt{\alpha_1}} \sin(\sqrt{\alpha_1}t) - \frac{2\alpha - \alpha_4}{2\alpha_4 \alpha_5 \sqrt{\alpha_2}} \sin(\sqrt{\alpha_2}t) \\
q_2(t) &= \frac{t}{3} + \frac{\alpha - \alpha_4}{2\alpha_3 \alpha_5 \sqrt{\alpha_1}} \sin(\sqrt{\alpha_1}t) - \frac{\alpha - \alpha_4}{2\alpha_4 \alpha_5 \sqrt{\alpha_2}} \sin(\sqrt{\alpha_2}t) \\
q_3(t) &= \frac{t}{3} + \frac{\alpha}{2\alpha_3 \alpha_5} \sin(\sqrt{\alpha_1}t) - \frac{\alpha}{2\alpha_4 \alpha_5} \sin(\sqrt{\alpha_2}t)
\end{align*}
\]  

(6.101)

The velocities are therefore given by:

\[
\begin{align*}
\dot{q}_1(t) &= \frac{1}{3} + \frac{2\alpha - \alpha_3}{2\alpha_3 \alpha_5 \sqrt{\alpha_1}} \cos(\sqrt{\alpha_1}t) - \frac{2\alpha - \alpha_4}{2\alpha_4 \alpha_5 \sqrt{\alpha_2}} \cos(\sqrt{\alpha_2}t) \\
\dot{q}_2(t) &= \frac{1}{3} + \frac{\alpha - \alpha_4}{2\alpha_3 \alpha_5 \sqrt{\alpha_1}} \cos(\sqrt{\alpha_1}t) - \frac{\alpha - \alpha_4}{2\alpha_4 \alpha_5 \sqrt{\alpha_2}} \cos(\sqrt{\alpha_2}t) \\
\dot{q}_3(t) &= \frac{1}{3} + \frac{\alpha}{2\alpha_3 \alpha_5} \cos(\sqrt{\alpha_1}t) - \frac{\alpha}{2\alpha_4 \alpha_5} \cos(\sqrt{\alpha_2}t)
\end{align*}
\]  

(6.102)

Our aim is now to calculate the instants \(t_1\) and \(t_2\) when the first and second balls separate, and when the second and the third balls separate, respectively. From the equation \(q_1(t_1) = q_2(t_1)\) one finds that \(t_1\) is the solution of the equation:

\[
\frac{3\alpha - 2\alpha_3}{\alpha_3 \sqrt{\alpha_3}} \sin(\sqrt{\alpha_1}kt_1) - \frac{3\alpha - 2\alpha_4}{\alpha_4 \sqrt{\alpha_4}} \sin(\sqrt{\alpha_4}kt_1) = 0
\]  

(6.103)

From the equation \(q_2(t_2) = q_3(t_2)\) one finds:

\[
\frac{1}{\alpha_3} \sin(\sqrt{\alpha_1}kt_2) - \frac{1}{\sqrt{\alpha_4}} \sin(\sqrt{\alpha_2}kt_2) = 0
\]  

(6.104)

It is noteworthy that the values of \(kt_1\) and \(kt_2\) do not depend on \(k\), since the coefficients of the equations in (6.103) and (6.104) are independent of \(k\). Let us analyze two cases separately, when \(t_1 < t_2\) and when \(t_1 > t_2\).
CHAPTER 6. GENERALIZED IMPACTS

Case 1: $t_1 < t_2$  This is the case when the balls 1 and 2 separate first. Then on a certain interval of time one gets $q_1 = 0$. From (6.102) and (6.103) the following is true

Claim 6.2  The velocity of the first ball after separation between the first and the second balls is independent of the stiffness $k$, and depends only on the ratio $\alpha$. The velocity of the second and third balls at the separation instant depends only on $\alpha$ also.

Then on $(t_1, t_1 + \delta)$ for a certain $\delta > 0$ the dynamical equations are:

\[
\begin{cases}
\ddot{q}_2(t) = \alpha k (q_3 - q_2) \\
\ddot{q}_3(t) = \alpha k (q_2 - q_3)
\end{cases}
\]  (6.105)

with the initial conditions $q_2(t_1)$, $\dot{q}_2(t_1)$, $q_3(t_1)$ and $\dot{q}_3(t_1)$. The solutions of this system are given by:

\[
\begin{align*}
q_2(t) &= \frac{q_2(t_1)+q_3(t_1)}{2} + \frac{\dot{q}_2(t_1)+\dot{q}_3(t_1)}{2} t + \frac{q_3(t_1)-q_2(t_1)}{2} \frac{1}{\sqrt{2\alpha k}} \sin(\sqrt{2\alpha k} t) \\
&\quad + \frac{q_3(t_1)-q_2(t_1)}{2} \cos(\sqrt{2\alpha k} t) \\
q_3(t) &= \frac{q_2(t_1)+q_3(t_1)}{2} + \frac{\dot{q}_2(t_1)+\dot{q}_3(t_1)}{2} t - \frac{q_3(t_1)-q_2(t_1)}{2} \frac{1}{\sqrt{2\alpha k}} \sin(\sqrt{2\alpha k} t) \\
&\quad - \frac{q_3(t_1)-q_2(t_1)}{2} \cos(\sqrt{2\alpha k} t)
\end{align*}
\]  (6.106)

One deduces the velocities

\[
\begin{align*}
\dot{q}_2(t) &= \frac{\dot{q}_2(t_1)+\dot{q}_3(t_1)}{2} + \frac{\dot{q}_3(t_1)-\dot{q}_2(t_1)}{2} \cos(\sqrt{2\alpha k} t) \\
&\quad + \frac{q_3(t_1)-q_2(t_1)}{2} \sqrt{2\alpha k} \sin(\sqrt{2\alpha k} t) \\
\dot{q}_3(t) &= \frac{\dot{q}_2(t_1)+\dot{q}_3(t_1)}{2} - \frac{\dot{q}_3(t_1)-\dot{q}_2(t_1)}{2} \cos(\sqrt{2\alpha k} t) \\
&\quad - \frac{q_3(t_1)-q_2(t_1)}{2} \sqrt{2\alpha k} \sin(\sqrt{2\alpha k} t)
\end{align*}
\]  (6.107)

From claim 6.2 we know that the velocities at $t_1$ do not depend on $k$. Now it can be calculated that

\[
\frac{q_3(t_1) - q_2(t_1)}{2} = \frac{\sqrt{2\alpha}}{4\alpha_5 \sqrt{k}} \left[ \frac{1}{\sqrt{\alpha_3 \sin(\sqrt{\alpha_3 k} t_1)}} - \frac{1}{\sqrt{\alpha_4}} \sin(\sqrt{\alpha_4 k t_1}) \right]
\]  (6.108)

The right-hand-side of (6.108) does not depend on $k$ but only on $\alpha$ since from (6.103) $k t_1$ depends only on $\alpha$. Let us now calculate the time $t_3$ when the second and the third balls separate (note that $t_3 \neq t_2$ in general). From (6.106) one finds that

\[
\tan(\sqrt{2\alpha k} t_3) = \frac{q_3(t_1) - q_2(t_1)}{\dot{q}_2(t_1) - \dot{q}_3(t_1)} \sqrt{2\alpha k}
\]  (6.109)
6.5. GENERAL RESTITUTION RULES FOR MULTIPLE IMPACTS

One deduces from (6.109) that $kt_3$ does not depend on $k$ but only on $\alpha$. Hence from (6.107) the following is true:

**Claim 6.3** Assume that after the first and the second balls have separated at $t_1$, then the second and third balls separate at $t_3$, and that the first ball evolves freely on $[t_1, t_3]$. Then the velocities of the second and of the third balls at $t_3$ depend only on the stiffnesses ratio $\alpha$.

**Case 2: $t_1 > t_2$** Similar calculations can be led in this case. One computes $t_2$ from (6.101) and finds from (6.102) that the velocities at $t_2$ depend only on $\alpha$, not on $k$. Then one obtains the instant $t_4$ when the first and the second balls separate, and concludes that $kt_4$ depends on $\alpha$, not on $k$. Hence the final velocities at $t_4$ depend on $\alpha$ only.

In summary, contrarily to the one-degree-of-freedom case treated in chapter 2, section 2.1, a simple sequence of approximating compliant problems for the 3-balls system does not possess a unique limit that represents the dynamical behaviour of the rigid body model. The central point is that the rigid model does not contain enough information on the physical properties of the system, to allow for a unique postimpact solution. Merely stating $k_1 = +\infty$ and $k_2 = +\infty$ is in fact a very vague manner of describing the contact-process. The approximating problem shows that there may be different sorts of infinity, or of rigidity. The idea would be to incorporate some *a priori* knowledge on the system (i.e. the ratio $\alpha$) to be able to choose a particular restitution rule. For instance if the three balls are made of the same material, then $\alpha = 1$. One can then suppose rigidity, and at the same time choose the restitution coefficients that correspond to this ratio.

From a mathematical point of view, let us recall that existence of solutions has been proved for nonsmooth $\partial \Phi$ and $T_L = 0$ in [417]. The 3-balls system exactly fits within this framework. It would be interesting to study the case $T_L < 0$, adding dampers to the springs.

**Remark 6.19** Note that we have not proved that there are no subsequent collisions between the balls after $t_3$ or $t_4$. In fact, numerical simulations [96] show that $q_1(t_1) < 0$ and that $\dot{q}_2(t) > 0$, $\dot{q}(t) > 0$ for all $t \geq t_1$. Therefore when $t_1 < t_2$, the velocities at $q_1(t_1)$, $q_2(t_3)$ and $q_3(t_3)$ are the final ones. When $t_1 > t_2$, the velocities at $q_1(t_4)$, $q_2(t_2)$ and $q_3(t_2)$ are the final ones. This is a very interesting fact since it means that one can study the process outcomes for the rigid case without having to integrate stiff ODE's. Indeed since only $\alpha$ plays a role, the only thing that is important is the stiffnesses ratio, not the absolute value of $k_1$ and $k_2$. Numerical results in [96] show that all the possible outcomes deduced from the rigid body analysis (see figure 6.10) correspond to at least one value of $\alpha \geq 0$ (see also the comments in subsection 6.5.6).
6.5.5 2-balls

We now come back briefly to the 2-balls system. Let us define \( \tilde{q}_1 = q_2 - q_1 \), so that the inertia matrix in new coordinates is given by \( \tilde{M} = \begin{pmatrix} m_1 & m_1 \\ m_1 & m_1 + m_2 \end{pmatrix} \). Since the preimpact velocities are \( \dot{q}_1(t_L^-) = 1 \) and \( \dot{q}_2(t_L^-) = 0 \), it follows that before the shock the system moves along the constraint \( f_2(\tilde{q}) = 0 \), i.e. \( \tilde{q}_1 = 2R \). Hence it is logical to apply the restitution rule along \( f_2(\tilde{q}) = 0 \). Now which restitution should we apply? First notice that the kinetic angle at the singular point between both constraints is \( \theta_{12} = \arccos \left( -\sqrt{\frac{m_1}{m_1 + m_2}} \right) \), thus \( \theta_{12} > \frac{\pi}{2} \). We deduce that motion is possible without imposing any tangential restitution. In fact it is easily seen that applying Newton's rule to \( \tilde{q}_{\text{norm},1} \) yields \( \tilde{q}_1(t_L^+) = -\tilde{q}_1(t_L^-) \) \( (T_L = 0) \), and leaving \( \tilde{q}_{\text{tang},1} \) continuous yields \( \tilde{q}_2 \) continuous also. Hence \( \tilde{q}_1(t_L^+) = -\tilde{q}_1(t_L^-) = -1 \) is the only feasible motion. This is exactly what is found using the algorithm in [187] after three iterations. We have however chosen an impact rule more or less arbitrary. The same type of operations that we performed on the 3-balls system can be applied to the 2-balls case. It then follows that an infinity of possible outcomes can occur (even with the restriction \( T_L = 0 \)), and it is not possible to choose arbitrarily among one of them.

6.5.6 Additional comments and studies

Newby [396] provides a very interesting study of the 3-balls problem, quite similar to the above one. He uses the same approximating compliant model, and derives an analytical solution. Then he studies numerically the influence of the stiffnesses ratio \( \alpha \) on the outcome. In particular he investigates whether it is possible to recover \( \alpha \) from the final velocities values (called in [396] the inverse scattering problem): in fact this is not always possible. He finds that all possible postimpact outcomes are described with compliant models such that \( \alpha \in (0, \alpha_{\text{max}}) \), for some \( \alpha_{\text{max}} \). There is therefore a periodicity in \( \alpha \) in the dynamics. In particular the solution \( \dot{q}_1 = -\frac{1}{2}, \dot{q}_2 = \dot{q}_3 = \frac{2}{3} \) occurs for an infinite number of \( \alpha \)'s, not only for \( \alpha = \infty \). He also finds that balls 1 and 2 separate at the same time as bodies 2 and 3 (a sort of symmetrical double collision with \( t_1 = t_2 \)) when \( \alpha = \frac{(3n^4 - 2n^2 + 3) + \sqrt{9n^8 - 12n^6 - 42n^4 - 12n^2 + 9}}{8n^2} \), \( n = 2, 3, 4 \ldots \). Such a condition may be investigated by equalling the coefficients of both equations in (6.105) and (6.104). Walkiewicz and Newby [567] deal with Newton's cradle with 2 or 3 balls. They use momentum and energy conservation equations to derive the possible outcomes. In the case of 2 balls, there is a unique solution, which is natural since this is a simple impact(12). In the 3-balls case, there are an infinity of solutions. They point out that this might be due to interaction forces between the balls, whose detailed knowledge should be known to describe which one of the solutions is the right one. In [194] it is mentioned that in an \( n \)-balls Newton's cradle problem, the "classical" solution (that is found in most of the

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12Recall that what we denoted as the 2-balls system is not a Newton's cradle, since there is an additional obstacle.
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textbooks [23] p.62, [439] p.52, [44] p.257, [580] p.48, and is given by \( \dot{q}_1(t_k^-) = \dot{q}_2(t_k^+) = 0, \dot{q}_3(t_k^-) = 1 \) is not a necessary consequence of momentum and energy conservation laws. The authors use a compliant approximating problem and study how deformations "cross" the chain via a vibrational analysis. They show that nonuniqueness of the outcome prevails for a given initial deformation. In [102] the big difference between a succession of simple impacts and multiple impacts is pointed out. The 3-balls compliant system is studied numerically. In [276] the author studies velocity and momentum gains in systems involving several simple collisions, with different masses of the objects, following experimental and numerical results in [189]. Then he applies the results to a chain of balls falling vertically on a rigid ground. Other works dealing with velocity amplification through collisions in a linear chain of spheres can be found in [188] [515] [513] [565]. This is related to the experimental evidence that when a chain of balls collides with the ground, then the highest one may undergo a very sudden acceleration, just as if all the others had transmitted all their energy to it. This is also related to studying the optimal mass ratio for maximum speed ratio (i.e. which masses should the balls have so that for an initial velocity of the first one, the last one has the maximum final velocity?) [251] [188]. Other earlier studies on Newton's cradle can be found in [309] [500] [103] [473] [488] [355] [356] [284].

It is noteworthy that most of the cited references have been published in the American J. of Physics, which is dedicated to Physics teachers. This shows that most of the interest to this problem has been for teaching purposes. In view of the simplicity of the experimental device, this is natural. But it is worth recalling that despite of its apparent simplicity, this collisions process contains all the ingredients that render it a complex dynamical system, i.e. codimension > 1 unilateral constraint, non-orthogonal constraints (see subsection 6.5.8). This is true either from the mathematical or from the mechanical point of view. But the same can be said of the rocking block system. It is simple, and at the same time as complex as Newton's cradle. It is likely that many people are afraid by the fact that the collision occurs through a surface or a line, not a point, and are not realizing that both systems are quite similar.

Mathematical studies on systems of \( n \)-balls moving on a line can be found in [586] [587], in close relationship with studies of billiards. The shocks are assumed to be elastic, and the system is submitted to a potential field (constant in [586], varying in [587]), hence only the conservative case is considered. The main goal of these works is the characterization of Lyapunov exponents [181] definition 5.8.2 (13) for such systems (14). The problem related to multiple collisions is taken care of in [586], where the corresponding flow with collisions (see chapter 1, section

\footnote{Roughly speaking, Lyapunov exponents are invariant numbers attached to a system that allow one to classify the attractors of the system. For instance, positive Lyapunov exponents guarantee chaotic behaviour.}

\footnote{Indeed it is known that the theory on which Lyapunov exponents rely is not yet applicable to nonsmooth dynamical systems [296]. It therefore represents a current challenge to study Lyapunov exponents for impacting dynamical systems.}
1.3) is defined in an almost-everywhere sense. The studies in [586] [587] therefore belong to the same class as those described in chapter 7, section 7.1.4, on existence and stability of periodic trajectories in impacting devices, except that instead of Floquet's multipliers one studies Lyapunov exponents. They aim at characterizing the dynamics of certain mechanical systems subject to a set of unilateral constraints. For instance in the 2-balls case, [586] proves the existence of a periodic trajectory and studies its stability.

**Remark 6.20** One may wonder if this convergence study could be generalized to more complex cases. It is in fact a nice result because it provides some physical insight to what a specific multiple impact corresponds to, and why it is fundamentally impossible to decide from the rigid body model which one over an infinity of outcomes is the right one. But the result can be obtained thanks to the simplicity of the dynamics (linear system, central collisions, no friction). It does not seem possible to derive a similar analysis for systems like bipede robots. The possible postimpact velocities and all the admissible restitution coefficients can still be obtained via an analysis as above, but it is difficult to analyze an approximating compliant problem. Some preliminary results for the compass gait example can be found in [96].

### 6.5.7 Summary of the main ideas

We now end this section by a little summary of the invariant facts encountered in an impact problem.

- **1)** A Lagrangian system submitted to unilateral constraints is a point moving in its configuration space, striking some hypersurfaces. These surfaces may be quite different from the "real-world" constraints.

- **2)** For frictionless constraints, the virtual work principle implies that the percussion vector is along the Euclidean constraints normal.

- **3)** The kinetic metric allows to separate clearly the discontinuous velocity components from the continuous ones. It also allows one to get rid of some generally accepted rules (like the conservation of momentum at impact, or the fact that tangential velocities are continuous), that must in fact be deduced from the dynamics.

- **4)** The domain \( \Phi \) in the configuration space where motion is possible possesses a boundary \( \partial \Phi \). \( \partial \Phi \) may be smooth or possess singularities, like cusps.

#### 4a) If \( \partial \Phi \) is smooth, then the normal direction and the tangential ones are well-defined at any point, and the impact problem can be solved by associating some restitution coefficients where needed.

#### 4b) If \( \partial \Phi \) has singular points, then such directions are not well-defined. Besides constraints such as 1) kinetic energy loss, ii) postimpact velocity in the
feasible motion domain and iii) the form of the percussion vector, one has to impose impact rules to render the problem solvable (i.e. find a unique set of postimpact velocities). Theoretically, it is possible to determine the set of all possible values of the restitution coefficients that yield postimpact velocities satisfying i), ii) and iii). Without any further requirement on the postimpact motion or/and on the interaction impulsion, they all represent possible solutions.

6.5.8 Collisions near singularities: additional comments

It is worth noting that the general method presented above only intends to determine postimpact velocities under some energetical and geometrical constraints. In a sense (as long as one accepts that rigid body model and constant restitution coefficients actually represent the reality), this provides an exhaustive set of admissible coefficients that yield possible postimpact velocities. Now a question is to know whether, given a certain process with rigid bodies, there is the possibility to choose one (or at least of finite number) of such coefficients. As we saw in chapter 5, this may be done by imposing some additional impact rules, like in the algorithms proposed in [187] [221] and most importantly the sweeping process dissipative model. This can also be done via the study of compliant approximating problems as the 2- and 3-balls cases prove. This is however a hard task in general. It is important however to recall that this concerns the cases when the system strikes the domain of constraints at a singularity. In general given an initial velocity \( \dot{q}(t_0) \), the set \( S_0 \) of initial conditions \( q(t_0) \) in the configuration space that yield a collision at a singularity (or a multiple impact [232]) is of zero measure (see the planar disk example in subsection 5.3.3). Hence one may legitimately argue that it is not so crucial to study collisions at the singularity, but near the singularity. In other words, if a trajectory is initialized arbitrarily close to \( S_0 \), then it strikes only one surface at the same time and one can study its behaviour applying classical restitution rules. Such a behaviour will generally consist of successive collisions with the surfaces that form the singularity \(^{15}\). As the disk example still shows, one cannot expect in general continuity of the trajectories with respect to initial conditions near \( S_0 \). In fact there are 2 problems related to singularities of \( \partial\Phi \): i) If the singularity is attained, how can we define the restitution rule at this point?, ii) If the system undergoes successive simple impacts on the hypersurfaces that constitute the singularity, is this multiple impact wellposed? For instance is there continuity of the trajectories with respect to initial data?

As we shall see both problems i) and ii) are in fact equivalent, in the sense that if the constraints satisfy orthogonality conditions, then one can bring a positive answer to both i) and ii).

\(^{15}\)This again is a big difference between simple and multiple impacts: even if the shocks against each surface are purely elastic \((e_1 = e_2 = 1)\) and with no dissipation during flight-times, there may be a finite accumulation point of collisions times \( t_k \).
In view of this, there may be two points that require further insight: given a certain generalized restitution rule, is it physically sound (apart from the energetical and geometrical considerations)? What is the behaviour of trajectories starting close to $S_0$? These points are cleverly discussed by Ivanov [232] and Kozlov-Treshchëv [292], and we reproduce mainly Ivanov’s arguments in the following.

**Orthogonality of the constraints**

Assume that a Lagrangian system is submitted to several unilateral constraints of the form $q_i \geq 0$, $i \in \{1, ..., m\}$. At a multiple impact (i.e. at a singularity), assume one applies the rule

$$q_{\text{norm},i}(t_k^+) = -e_i q_{\text{norm},i}(t_k^-) \quad (6.110)$$

for the constraints $i \in \mathcal{I}_a \subseteq \{1, ..., m\}$ which become active at $t = t_k$ ($q_i(t) > 0$ for $t < t_k$, $q_i(t_k) = 0$), whereas

$$q_{\text{tang}}(t_k^+) = q_{\text{tang}}(t_k^-) \quad (6.111)$$

This corresponds to defining $n_q$ from the attained surfaces, whereas $t_q$ completes the orthonormal (in the kinetic sense) basis. It is argued in [232] that such a rule is physically coherent only if the inertia matrix is such that

$$e_i^T M^{-1}(q) e_j = 0 \quad (6.112)$$

for all $i, j \in \mathcal{I}_a$, $e_k$ is the $k$-th unit vector equal to $\nabla q f_k(q)$ in this case. The underlying idea is that if the conditions in (6.112) are satisfied, then the generalized percussion vector component $p_{q,i}$ has no influence on the jump in $q_{\text{norm},i}$, and vice-versa ($p_{q,j}$ and $q_{\text{norm},i}$). Indeed note from (6.6) that

$$\sigma_{q_{\text{norm}}} = n_q^T p_q$$

$$\sigma_{q_{\text{tang}}} = 0 \quad (6.113)$$

with $q_{\text{norm},i} = \frac{e_i^T q}{\sqrt{m_{ii}^{-1}}} = \frac{q_i}{\sqrt{m_{ii}^{-1}}}$, $t_{q,k} = e_k$; $m_{ii}^{-1}$ is the $i$-th diagonal element of $M^{-1}(q)$. Developing the first set of equations in (6.113) one gets

$$n_q^T p_q = \begin{bmatrix} \frac{e_i^T q}{\sqrt{m_{ii}^{-1}}} (p_{q,i} e_i + p_{q,j} e_j) \\ \frac{e_j^T q}{\sqrt{m_{jj}^{-1}}} (p_{q,i} e_i + p_{q,j} e_j) \end{bmatrix} \quad (6.114)$$

where Ivanov’s arguments clearly appear (Recall $p_q \in N(q)$, hence $p_q = p_{q,i} e_i + p_{q,j} e_j$ for some scalars $p_{q,i}$ and $p_{q,j}$). The conditions in (6.112) correspond to having the constraint surfaces mutually orthogonal in the kinetic metric sense. Let us recall that the following equivalences hold:

$$n_{q,i}^T \nabla_q f_j(q) = 0 \iff \nabla_q f_i(q) M^{-1}(q) \nabla_q f_j(q) = 0 \iff n_{q,i}^T M(q) n_{q,j} = 0 \quad (6.115)$$
The last equality clearly states orthogonality of both hypersurfaces $f_i(q) = 0, f_j(q) = 0$, in the kinetic metric sense. Note also that orthogonality just means that the vectors $n_{q,i}$ are equal to the vectors $t_{q,j}$, i.e. the normal vectors to the constraints $i$ are the tangential vectors to the constraint $j$.

To sustain his claim, Ivanov chooses a classical approximating problem for the disk striking an angle example. Both surfaces of the angle are modeled by a spring+damper system (see chapter 2), i.e. if $N_i$ denotes the normal interaction force and $x_i$ the surface normal deformation (recall that in this subsection we deal with frictionless constraints only) we get $N_i = k_n x_{n,i} + f_n x_{n,i}$ if $x_{n,i} > 0$, $N_i = 0$ otherwise. We emphasize by the subscripts $n$ that this represents a sequence of approximating problems, just as we have done in chapter 2. It is illustrated by simulation results for the approximating problem that the disk postimpact velocity and position are always different from the ones predicted by the above particular restitution rule. However the convergence of this approximating sequence towards the rigid case is not studied. This is easily understandable from the results presented in chapter 2, which show how complex such mathematical studies are (see problems 2.1, 2.2 and 3.1, see also the existential results for the sweeping process in subsection 5.3.3).

On the other hand, notice that the orthogonality condition is quite natural, since it implies that the dynamical equations in (6.113) are decoupled, i.e. are composed of one degree-of-freedom subsystems. As we saw in the introduction of section 6.5, the restitution rule in (6.110) (6.111) then makes sense.

**Remark 6.21** We have seen in chapter 2 that the most advanced existential results concerning systems with codimension $\geq 2$ unilateral constraints are restricted to $T_L = 0$, see [417] chapter 1. It is quite possible that under the additional condition of orthogonality of the constraints this extends to $T_L < 0$. This would be very nice and useful for control purposes, since it is much more realistic to assume $T_L < 0$. Moreover this is needed to obtain finite time stabilization on the constraint (in the case of a simple impact on one of the surfaces only).

**Remark 6.22** The orthogonality condition for the rocking block (see sections 6.4.1 and 6.5.2) can be written as

$$\begin{pmatrix}
0, \frac{2}{L}, 1
\end{pmatrix}
\begin{pmatrix}
\frac{1}{m} & 0 & 0 \\
0 & \frac{1}{m} & 0 \\
0 & 0 & \frac{1}{L}
\end{pmatrix}
\begin{pmatrix}
0 \\
1 \\
-\frac{L}{2}
\end{pmatrix}
$$

from which it follows that the cusp singularity in figure 6.5 has orthogonal tangent vectors if and only if

$$l = \frac{\sqrt{3}}{2} L$$

(6.117)
Hence if the dimensions of the block satisfy (6.117), the generalized restitution rule as defined in (6.110) and (6.111) possesses a theoretical meaning as we discussed above. It would be very interesting to validate experimentally such restitution law for a block satisfying (6.117), and to compare the results with those obtained when (6.117) is not satisfied. Notice that rocking cannot be described by a restitution rule as in (6.76) (6.77) if the constraints are orthogonal. Indeed assume the block rotates around corner A and strikes the base at B. Then \( q_{\text{norm.2}}(t_k^-) = 0 \), so that necessarily \( q_{\text{norm.2}}(t_k^+) = 0 \) as well. Hence either corner B rebounds (if \( c_2 > 0 \)), or the block stops its motion (\( c_2 = 0 \)). A generalized tangential restitution is useless since anyway, \( n q_{\text{norm}} \) is tangent to \( f_j(q) = 0 \), \( i \neq j \). One can easily visualize this situation in the plane, with the Euclidean metric. Hence one has to choose another set of vectors to define the basis in which the velocity is expressed, to be able to modify via restitution coefficients the direction of the velocity. We have done such choices for the 3-balls example above. Concerning this last system, let us investigate whether the constraints can or cannot be orthogonal, depending on the masses of the balls. The condition in (6.115) become

\[
\nabla_q f_1(q) M_q^{-1} \nabla_q f_2(q) = 0
\]

(6.118)

with \( f_1(q) = q_2 - q_1 \), \( f_2(q) = q_3 - q_2 \), and \( M_q = \text{diag}(m_1, m_2, m_3) \). Then one finds that orthogonality holds if and only if \( -\frac{1}{m_2} = 0 \), i.e. \( m_2 \) is infinite. In other words, the second ball is replaced by a fixed rigid obstacle. Then indeed the problem becomes trivial. This shows that in practice, a problem as simple as the 3-balls problem always involves a non-orthogonal multiple impact when the second and the third ball are in contact at the shock instant.

**Behaviour of trajectories close to singularities**

To study the behaviour of the trajectories initialized close to the set \( S_0 \), Ivanov analyzes the case of a 2-degree-of-freedom system submitted to 2 unilater constraints. They are assumed to form a singularity, hence it is not possible to suppose that they can be put in the form of a single constraint. The restitution coefficients for each constraint surface are chosen equal one to each other. Then a nonsmooth transformation (see subsection 1.4.2) is applied to study the system’s dynamics. The change of coordinates is such that the constraints are simply expressed in the new coordinates as \( q_1 \geq 0, q_2 \geq 0 \). Hence in the configuration space, one obtains a particle striking an angle. The angle value is assumed to be \( \alpha \). In view of the foregoing discussion, it is supposed that \( \alpha \neq \frac{\pi}{2} \) (in the kinetic metric sense). What is expected is that the system (the particle) is going to collide with one of the sides of the angle, and then start a sequence (finite or infinite, it is in fact the goal of the study to determine this) of rebounds. This is illustrated in figure 6.11, where it clearly appears that the number of impacts depends on the angle \( \alpha \) made by the 2 surfaces of constraint (the motion is that of a point-mass system).

To render the analysis possible, it is further assumed that the flight-times are zero. Hence the system evolves only under the influence of the impact map. The
smooth dynamics are neglected (In the language of remark 1.3.2, the flow with collisions reduces to the diffeomorphism). Finally Ivanov supposes that the inertia matrix of the system is such that $e_i^T M(q) e_i = 0$ for all $j \neq 1, 2$, $i = 1, 2$, so that impacts do not influence nonconstrained generalized position components. In view of these assumptions, one can obtain a relationship that relates the successive angles of incidence $\beta_k$ of the particle when it strikes one of the surfaces as

$$\beta_{k+1} = \alpha + \arccotan(e^{-1}\cot\beta_k)$$

with $\cot(x) = \frac{1}{\tan(x)}$, and $\arccotan(x) \circ \tan(x)$ is the identity. The relationship in (6.119) is thus the first-return map of the system, in terms of the angle of incidence of the trajectories. Several outcomes for the multiple impact process are then deduced from the behaviour of the map in (6.119). It is possible that the collision ceases if $\beta_n > \pi$ after $n < +\infty$ iterations. Then the system goes outside the constraint-well. When the map has a fixed point $\beta^*$, it may happen that for a set of initial conditions, the sequence $\{\beta_k\}$ converges towards $\beta^*$. Then the impact is similar to an inelastic impact (recall that it is assumed that flight times are of zero duration). Note that this may occur even though the surfaces are considered with a nonzero restitution coefficient. When the coefficients of restitution of both surfaces are not chosen equal, it can as well be concluded that only two outcomes are possible: either a bounded number of repetitive impacts, or an impact of arresting nature. It is shown in [232]...
that the case of an arresting impact occurs if the condition

\[ 2 \tan(\alpha) \leq \frac{1}{\sqrt{e}} (1 - e) \]  

(6.120)
is satisfied. The singularity is then a trap within which the system falls. If this condition (6.120) is not satisfied, the system behaves like if the multiple impact was a simple elastic impact.

In summary, there are two main types of double-impact (i.e. collisions near a codimension 2 singularity):

- When the constraints are orthogonal, one can use the restitution rule in (6.110) and (6.111).
- For nonorthogonal constraints, the stated two possible outcomes occur.

To conclude, notice that this multiple-impact analysis complicates drastically when the singularity is of codimension > 2. It is conjectured in [232] that as the number of constraint surfaces increases, the multiple collision process becomes stochastic.

Remark 6.23 This conclusions (obtained however under certain basic assumptions like neglecting the flight-times) tend to show that one should treat collisions at singularities with great care: indeed the postimpact motion does not depend directly on the separated properties of each surface. It is the result of a much more complex process of repeated (and accumulated in the sense that they are supposed to occur at the same time) collisions. Such results are a first step towards the determination of impact rules at singularities. Note that the sweeping process dissipative formulation and the conclusions in [232] are in accordance, in the sense that Ivanov concludes that a generalized arresting collision is a realistic outcome. Such postimpact motion is the result of the sweeping process rule when a particle strikes an angle whose kinetic value is \( \leq \frac{\pi}{2} \).

The multiple impact problem is also treated in great mathematical detail in the book by Kozlov and Treshchëv [292] Introduction, §12. The authors make extensive use of the so-called Coxeter group \( W \), i.e. the group of orthogonal transformations of \( \mathbb{R}^n \) (in the kinetic metric), generated by reflections relative to the surfaces of constraints where the multiple impact occurs. The terminology used in [292] is to call the half-spaces of possible motion determined by each constraints \( \text{symplicial cones} \), and their intersection a \( \text{chamber} \) (more or less equivalent to the tangent cones defined in convex analysis approaches like the sweeping process). When the constraints are defined with hyperplanes, a chamber becomes a \( n \)-cell [464] of \( \mathbb{R}^n \) (a cell is the generalization of a rectangle). It is possible to approximate locally the singularities by such cells by expanding the functions \( f_i(q) \) in their Taylor series,
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so that locally the constraints are given by a set of inequalities $c_i^T q \geq 0$, with $c_i = \nabla_q f_i(q)$. A billiard is called regularizable if the solution $q(t, q_0, \dot{q}_0)$ of the dynamical problem depends continuously on the initial data $(q_0, \dot{q}_0)$. For the case of elastic impacts, regularizability holds whenever the faces which constitutes the singularity make angles $\frac{\pi}{n}$, $n \in \mathbb{N}^*$ one with each other. In the planar case it can be seen that the number of impacts that occur during the multiple collision when the system strikes one of the two faces of the angle, is equal to $n$. This is called the multiplicity of the multiple impact. Several cases are illustrated in figures 6.12.

The following theorem is true, which once again confirms the above developments:

**Theorem 6.1 ([292])** If the surfaces of constraints occurring in a multiple impact are pairwise orthogonal (in the kinetic metric), then the billiard defined in its chamber is regularizable.

### 6.6 Constraints with Coulomb friction

We now examine what happens when there is some dry Coulomb friction at the contact point between the two colliding bodies (or systems). Following [54] we thus introduce an impulse coefficient $\mu$ between the tangential and the normal force components. The goal is to use the coordinates in (6.6) to explain why in general the addition of friction complicates much the problem. In particular this allows to understand easily why there has been so many studies devoted to the energetical consistency of such a contact problem (see chapter 4 for details).

The dynamical equations in (6.6) seem quite appealing to generalize the case of a particle striking an obstacle with friction to the $n$-dimensional case of a kinematic chain with unilateral constraints. Indeed, imagine that the generalized interaction force can be written as

$$ p_q = p_{q,n} M(q)n_q + \sum_{i=1}^{n-1} p_{q,t,i} M(q)t_{q,i} $$

---

**Figure 6.12:** Multiplicity in a multiple impact.
and that

$$p_{q,t,i} = \mu_i p_{q,n} \quad (6.122)$$

for some coefficients $\mu_i, 1 \leq i \leq n - 1$, that are the generalization of the two tangential directions in a 3-dimensional problem where one may associate one impulse ratios for each direction [58]. Then using (6.6) we should get $\sigma_{\text{norm}} = p_{q,n}$ and $\sigma_{\text{tang}} = \sum_{i=1}^{n-1} \mu_i p_{q,n}$. If $\dot{q}_{\text{norm}}(t_k^+) = -(1+e)\dot{q}_{\text{norm}}(t_k^-)$, then $\sigma_{\text{tang},i} = -(1+e)\dot{q}_{\text{norm}}(t_k^-)\mu_i$. Thus we should obtain

$$T_L = \frac{1}{2}(e^2-1)(\dot{q}_{\text{norm}}(t_k^-))^2 + \frac{1}{2}(1+e)^2(\dot{q}_{\text{norm}}(t_k^-))^2 \sum_{i=1}^{n-1} \mu_i - (1+e)\dot{q}_{\text{norm}}(t_k^-) \sum_{i=1}^{n-1} \mu_i \dot{q}_{\text{tang},i}(t_k^-) \quad (6.123)$$

Equation (6.123) generalizes equation (19) in [55] (see also [174]) for two particles colliding. If (6.123) and the above developments were true, then all the material and analysis for two particles (or spheres) colliding could generalize to any complex Lagrangian system submitted to unilateral constraints. Unfortunately, this is not the case in general as the following example proves. Obviously, this had already been noted by many authors, see e.g. [55], and is a common remark. But in the body of our developments we illustrate it through the use of the kinetic metric. In the same way, if we directly apply a tangential restitution rule with coefficient $\epsilon_t$ to $\dot{q}_{\text{tang}}$, we get (6.78). Does this make sense in general? i.e. is there a correspondence between the application of such a coefficient at the contact point along the "ambient" 2 or 3-dimensional Cartesian space, and its application along the tangent direction in the configuration space? Certainly this is not the case.

In the next example, it is shown that applying local rules does not in general lead to the global rules described in (6.121) and (6.122), if the vectors $t_q$ are chosen to be orthogonal, so that the kinetic energy expression is quite simple. Note that we already noticed problems related to the choice of an orthonormal basis $(n_q, t_q)$ when we analyzed feedback control of the impact phase above (see remarks 6.1 and 6.11). There we discussed about the integrability of the transformed velocity $\dot{q}_{\text{norm}}$ and $\dot{q}_{\text{tang}}$.

### 6.6.1 Lamina with friction

We shall consider again the benchmark example of the planar lamina striking a plane. In the Cartesian 2-dimensional plane, the percussion vector is given by $P = \begin{pmatrix} \mu p_n \\ p_n \end{pmatrix}$ when $P \in \partial C$ (sliding regime). Thus $p_q = J^TP = \begin{pmatrix} \mu \\ 1 \\ \mu h(\theta) - f(\theta) \end{pmatrix}p_n$. $J$ is given in (6.16) (Compare with (6.121) for the expression of the percussion vector without Coulomb friction). Can we decompose the percussion vector as $p_q = p_{q,n} + p_{q,t}$ such that $n_q^Tp_{q,t} = 0$ and $t_q^Tp_{q,n} = 0$? The part of $p_q$ that is along the euclidean normal to the constraint is $p_{q,n} = \begin{pmatrix} 0 \\ p_n \\ -f(\theta)p_n \end{pmatrix}$. Then the rest of the
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The percussion vector is given by $p_{q,t} = \begin{pmatrix} 1 \\ 0 \\ h(\theta) \end{pmatrix} \mu p_n$. Obviously we have $t_q^T p_{q,n} = 0$, but simple calculations yield $n_q^T p_{q,t} = \mu p_n \frac{h(\theta) H(\theta)}{\sqrt{m I}} \frac{m I}{h(\theta)}$ that is not zero in general. Also $t_q^T p_{q,t} = \frac{\mu p_n}{\sqrt{m H^2(\theta) + I}} \left( \begin{array}{c} 1 \\ \frac{1}{h(\theta)} \end{array} \right)$. The conclusion is that there is in fact a "merging effect" between the frictional components and the normal ones, such that friction influences both parts of the dynamical equation in (6.6). The problem is that we can express $p_{q,t} = F(\mu, \theta)p_{q,n}$, but in general $n_q^T F(\mu, \theta)p_{q,n} \neq 0$. This may be possible if $h(\theta) = 0$ (the lamina is a particle or the center of gravity at the impact time is on the boundary) or if $H(\theta) = 0$ (i.e. $h(\theta)$ is a constant, e.g. the lamina is a disk). Thus the only way for the frictional and normal parts to be "decoupled" (i.e. the friction influences only the second equation in (6.6)) is that the lamina is a disk. We do not pursue here the calculations that would allow to express $T_L$ as a function of $e$, $\mu$ and preimpact velocities. As we noted previously, such energy loss calculations are in general very lengthy.

Remark 6.24 Let us now come back on what we discussed in remark 6.10. In that case we had seen that the frictionless impact equations could be written as in (6.47). What happens with dynamic friction? Writing down dynamical equations as in (6.19) (6.20) (6.21) we obtain

\begin{align*}
    m \ddot{a} - mF(\theta)\dot{a} &= \mu p_n \quad (6.124) \\
    m \dot{b} - mH(\theta)\dot{b} &= p_n \quad (6.125) \\
    -mF(\theta)\dot{a} - mH(\theta)\dot{b} + (mF^2(\theta) + mH^2(\theta) + I)\dot{\theta} &= (\mu h(\theta) - f(\theta) - F(\theta)\mu - H(\theta))p_n \quad (6.126)
\end{align*}

Then it can be calculated that

\begin{align*}
    p_n &= \frac{m I}{I + m H(\theta)(\mu h(\theta) - f(\theta))} \sigma_b(t_k) \quad (6.127) \\
    \sigma_\theta(t_k) &= \frac{m(\mu h(\theta) - f(\theta))}{I + m H(\theta)(\mu h(\theta) - f(\theta))} \sigma_b(t_k) \quad (6.128) \\
    \sigma_a(t_k) &= \frac{\mu I + \mu m F(\theta)h(\theta) - m F(\theta)f(\theta)}{I + m H(\theta)(\mu h(\theta) - f(\theta))} \sigma_b(t_k) \quad (6.129)
\end{align*}

It clearly appears from (6.129) that, for a given configuration, the frictionless process may not imply any tangential velocity $\dot{a}$ reversal, whereas the dynamic friction case may imply it, depending on the value of $\mu$. Indeed assume $\mu = 0$. Then $\dot{a}(t_k^+) = \dot{a}(t_k^-) - \frac{m F(\theta) f(\theta)}{I + m H^2(\theta)}$. It is quite possible that for a given $\theta$, $\text{sgn}(\dot{a}(t_k^+)) \neq \text{sgn}(\dot{a}(t_k^-))$. For the same value of $\theta$, one may have $\text{sgn}(\dot{a}(t_k^+)) = \text{sgn}(\dot{a}(t_k^-))$ if $\mu > 0$. Recall the crucial role of such tangential velocity reversal in the possible gain of kinetic energy.
at impacts. Now using (6.127) (6.128) (6.129) we can rewrite (6.47) as

\[
\mathcal{M}(\theta, \mu) \begin{pmatrix} \sigma_a \\ \sigma_b \\ p_n \end{pmatrix} = \mathcal{H}(\theta, \mu) \begin{pmatrix} \sigma_{\dot{a}} \\ \sigma_{\dot{b}} \\ p_{\dot{n}} \end{pmatrix}
\]  

(6.130)

Although \(\text{rank}(\mathcal{M}(\theta, 0)) = 3\), it is easily verified that \(\det(\mathcal{M}(\theta, \mu_c)) = 0\) for \(\mu_c = \frac{l + m f^2(\theta)}{m f(\theta) h(\theta)}\). In fact we have:

\[
m f(\theta) h(\theta) - m f(\theta)
\]

(6.131)

and

\[
\mathcal{M}(\theta, \mu) = [I + m H(\theta)(\mu h(\theta) - f(\theta))] I_3
\]

(6.132)

In the case \(\mu = \mu_c\) one thus arrives at a kind of undeterminancy (different from what we discussed in subsection 5.4.2, where we dealt with configurations with zero relative normal preimpact velocities): it is not possible to obtain a unique value of the vector \(\sigma_0\) from (6.130). This means that additional collision rules have to be defined. For instance, Moreau’s algorithm [381] (see subsection 5.4.2, see also example 4.1) deals with inelastic shocks \(b(t_\pi^-) = 0\) and yields a unique solution in case of nontangential impact \(b(t_\pi^+) < 0\). It is noteworthy that the critical value of the Coulomb friction coefficient \(\mu_c\) is exactly the one such that \(B(\theta, \mu_c) = 0\) in 5.83, subsection 5.4.2.

\[\nabla\nabla\]

Let us note that following [139], a good choice of the tangent vectors \(t_q\) is \(t_{q,1}\) as above and \(t_{q,2} = \begin{pmatrix} -h(\theta) \\ f(\theta) \\ 1 \end{pmatrix}\) (we omit the normalization). It can be verified that \(t_{q,2}^T M(q) n_q = 0\). It is not obvious which advantages this choice may bring compared to any other one for our purposes.

**Remark 6.25** The complications introduced by Coulomb’s dynamical friction can be given the following interpretation [139]: in the configuration space, the *generalized friction cones* defined from the real-world ones by applying suitable coordinate
transformations, are not as "nice" as the real-world friction cones. In fact, they are not in general symmetric with respect to the generalized normal direction. Even more, they can dip below the tangent plane. This is a way to explain the strange behaviours we discussed in subsection 5.4.2. Incidentally, Erdmann [139] considers the same example as in [381] to illustrate the validity of such ambiguous situations. When one applies a torque that tends to make the chalk detach from the surface, frictional effects can compensate for it so that the chalk remains in contact with the board.

**Remark 6.26 About Stronge's energetical coefficient**

Let us consider the energetical coefficient as defined in (4.47). Although Stronge analysis is done basically for a compliant environment, his coefficient is energetical and we can define it for the rigid case as

\[ e_*^2 = \frac{T_n(t_k^+)}{T_n(t_k^-)} \]  

(6.133)

\( T_n \) denotes the part of the kinetic energy due to normal velocity at the contact point. In the lamina example case, we can separate the kinetic energy into \( T_n \) and \( T_t \), where \( T_n = T(0, b, 0) \). If we introduce it into (6.133) we get \( \dot{b}^2(t_k^+) = e_*^2 \dot{b}^2(t_k^-) \). We retrieve Newton's rule. This is also the case if we work with the transformed coordinates in (6.6). We then obtain \( \dot{q}_{\text{norm}}^2(t_k^+) = e_*^2 \dot{q}_{\text{norm}}^2(t_k^-) \). Notice that this is independent of friction. In the frictionless case, it is concluded from the total kinetic energy loss at \( t_k \) that \( e_* \leq 1 \). When \( \mu \neq 0 \), we have

\[ 2TL(t_k) = (e_*^2 - 1)(\dot{q}_{\text{norm}}(t_k^-))^2 + \dot{q}_{\text{tang}}(t_k^+)^T \dot{q}_{\text{tang}}(t_k^+) - \dot{q}_{\text{tang}}(t_k^-)^T \dot{q}_{\text{tang}}(t_k^-) \]

Using the same choice of vectors \( n_q \) and \( t_q \) as in (6.29) and (6.30), one obtains

\[ \sigma_\text{tg} = \begin{pmatrix} \frac{-\mu q p}{\sqrt{m}} \\frac{-1}{\sqrt{mI(-H(\theta)+1H(\theta)-f(\theta))}p} \end{pmatrix} \]  

(6.134)

where

\[ p_i = -\sqrt{mH^2(\theta) + I(1 + e_*)(\dot{q}_{\text{norm}}(t_k^-))^2} \frac{\sqrt{mI}}{I + mH(\theta)(\mu H(\theta) - f(\theta))} \]  

(6.135)

Although we will not develop further the expression of \( TL \), it is really not clear from (6.134) and (6.135) why \( e_* \) should be \( \leq 1 \) when friction is present. This does not seem provable via the constraint \( TL \leq 0 \).

Until now we have presented and discussed the nature of solutions, macroscopic models of impacts and problems related to the generalization to multiple-bodies systems. In the next chapter we shall discuss the extension of Lyapunov techniques to such systems for stability analysis purposes.
Chapter 7

Stability of solutions of impacting systems

In chapter 1 we have discussed about the nature of solutions of dynamical problems involving impulsive impacts, and some of their properties like existence, uniqueness, continuous dependence on initial conditions and parameters. Roughly, we have seen that those dynamical systems are represented by measure differential equations (MDE). We have also discussed the differences between dynamical equations of systems with unilateral constraints and MDE's. In view of some important applications like for instance control of manipulators subject to impacts, or simply the study of general impacting mechanical systems, it is important to study stability of solutions of those differential equations. In this chapter we start by presenting stability concepts which are the extension of Lyapunov stability to MDE's (as in section 1.2). Then we focus on the stability of impact Poincaré maps. In chapter 2 we have seen that under reasonable assumptions the solutions of compliant approximating problems converge to those of the limit rigid problem. This motivates us to study the relationship between stability of solutions of compliant approximating problems and stability of solutions of the rigid limit problem: in other words, we study how a particular stability property evolves when the stiffness becomes infinite.

7.1 General stability concepts

7.1.1 Stability of measure differential equations

Firstly, let us consider measure differential equations $\dot{z} = f(z, t, Du)$ and let us study the behaviour of the solution $z(t)$ for $t \geq t_0$. As we have seen in chapter 1, subsection 1.2, if $u$ is of bounded variation, then $z(\cdot)$ is $RCLBV$ on closed time intervals (provided of course that $f(\cdot, \cdot, \cdot)$ satisfies the requirements explained in chapter 1, subsection 1.2), with possible discontinuities at times $t_k$, $k \geq 0$. Few works in the western mathematical literature have been devoted to this problem.
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[307] [452] until recently (especially with the work of Lakshmikantham and his co-
workers [316] [327]). A thorough treatment and many references from the east side can be found in [27]. Basically, one has to analyze the system during the continu-
ous motion on intervals \((t_k, t_{k+1})\) and at discontinuities \(t_k\). Lyapunov functions \(V\) derivatives have to be understood as \(V_{rd} = \lim_{h \to 0^+} \sup_t \frac{V(t+h)-V(t)}{h}\) (the upper right Dini derivative of \(V\)) almost everywhere (assuming the solutions are RCLBV), and \(V(t_k) = \sigma_V(t_k)\delta_{t_k}\), applying a generalized chain rule [562] at impact times \(^1\). Lyapunov's second method extends by stating \(V_{rd} \leq 0\) and \(\Delta V(t_k) \leq 0\), [27] chapter 13. This is done under certain conditions on the jump times: in particular it is assumed that \(t_k < t_{k+1}\) for all \(k \geq 0\), and that \(t_k \to +\infty\) as \(k \to +\infty\). The stability definitions are also given for the fixed point of the smooth dynamical equations. For the sake of completeness, we provide the definitions of Lyapunov stability stated in [27].

Let a system be given by the following set of dynamical equations

\[
\begin{align*}
\dot{x} &= f(x, t), & t &\neq t_k(x) \\
\sigma_x(t_k) &= I_k(x(t_k)), & t &= t_k(x)
\end{align*}
\]  

(7.1)

Let the following conditions be satisfied

- The function \(f(x, t)\) is continuous, \(f(t, 0) = 0\), and is Lipschitz continuous.
- The functions \(I_k\) are continuous and \(I_k(0) = 0\) for \(k \in \mathbb{N}\)
- There exists a constant \(h < +\infty\) such that if \(||x|| \leq h\), then \(x + I_k(x) < +\infty\)
- The discontinuities times of any solution \(x(t)\) verify \(0 = t_0 < t_1(x) < t_2(x) < \ldots, \lim_{k \to +\infty} t_k(x) = +\infty\). The functions \(t_k(\cdot)\) are continuous.

Definition 7.1 (Lyapunov stability for MDE [27]) The solution \(x(t)\) of sys-
tem (7.1) is stable if for all \(\varepsilon > 0\), for all \(\eta > 0\), for all \(\tau_0 \geq 0\) such that \(|\tau_0 - t_k| > \eta\), there exists a \(\delta > 0\) such that for all \(x_0 \in \mathbb{R}^n\) with \(|x_0 - x(t_0)| < \delta\), for all \(t \in J^+(\tau_0, x_0)\) with \(|t - t_k| > \eta\), then \(|\varphi(t; t_0, x_0) - x(t)| < \varepsilon\).

The instants \(t_k\) denote the discontinuity instants of the solution \(x(t)\). The set \(J^+(\tau_0, x_0)\) denotes the maximal interval of existence of the solution \(\varphi(t; t_0, x_0)\) with initial conditions \((\tau_0, x_0)\). Similar definitions can be adapted to uniform, asymptotic stability, as in the classical non-impulsive case. We do not reproduce them here for the sake of brevity. Note that in general (when the \(t_k\)'s are not fixed but depend on the state), two different solutions \(x(t)\) and \(\varphi(t)\) will possess discontinuities at different times. Hence it is not possible to require that in a neighborhood of the times of discontinuities of \(x(t)\), the two solutions are arbitrarily close one to each

\(^1\)Notice that in general we do not have \(\dot{V} = \frac{\partial V}{\partial x} (\{\dot{x}\} + \sigma_x \delta_{t_k})\), but we do have \(\dot{V} = \{\dot{V}\} + \sigma_V(t_k)\delta_{t_k}\).
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other. However two solutions that start very close one to each other (in a smooth dynamics interval) are required to be close outside a (time) neighborhood of the jump instant. This is the role played by \( \eta \) in definition 7.1: if one takes a small enough \( \delta \), then one can look at both solutions outside an interval \((t_k - \eta, t_k + \eta)\) and conclude that on this interval they are close one to each other. One may take an arbitrarily small \( \eta \) by imposing an arbitrarily small \( \delta \). For the case of \( x(t) = 0 \) (note that \( x = 0 \) is a fixed point of the system (7.1)), one can define Lyapunov stability in the classical way [316]. This is also the case if the \( t_k \)'s are fixed, since then all solutions have discontinuities simultaneously.

Lyapunov second method extends to such systems. In fact the following theorem is true

**Theorem 7.1 (Lyapunov second method for MDE [27])** Let the above conditions be satisfied, and let functions \( V(t, x) \) and \( \alpha \) of class \( K \) exist with

\[
V(t, 0) = 0 \quad \text{for all } t \geq t_0\]

such that

\[
\begin{align*}
\alpha(||x||) & \leq V(t, x) \quad \text{for all } t \geq t_0 \text{ and } x \in \mathbb{R}^n \\
\dot{V}_{rd}(t, x) & \leq 0 \quad \text{for all } t \neq t_k(x) \\
\sigma_V(t_k) & \leq 0 \quad \text{for } t = t_k(x)
\end{align*}
\]  

(7.2)

Then the solution \( x = 0 \) of the system (7.1) is stable in the sense of definition 7.1.

\[\nabla\nabla\]

Note that in accordance with the last condition on the jump times, Lyapunov functions are defined as being piecewise continuous. In the same manner, theorem 7.1 can be extended to guarantee uniform asymptotic stability. It clearly appears that definition 7.1 requires that the continuous part of the dynamics be Lyapunov stable (see the second condition in (7.2)), as well as the discrete dynamics, i.e. the mapping which associates to preimpulse values a postimpulse value of the state.

**Remark 7.1** Notice that since Lyapunov techniques extend to such systems, passivity (or dissipativity) definitions and tools for continuous [197] [198] as well as discrete time [86] systems should also be considered to characterize systems as in 7.1. Basically, one should consider the dissipativity properties of both the impulsive map in (7.1) (i.e. the impact map for our mechanical systems) and the flow of the system between the collisions (i.e. in one shot dissipativity of the flow with collisions, see remark 1.3.2). Note that for mechanical systems with unilateral constraints, the properties of the impact map are independent of the applied controller. On the contrary, the properties of the smooth flow between impacts can be modified via suitable feedback control.
Additional comments and studies

Liu [316] extends the definition of stability of measure differential equations in the spirit of [27], and considers stability in terms of two measures (Here the word measure has not the common meaning of a measure as a function from a set of subsets into $[0, +\infty)$ [178]. It refers to functions $h(t, x)$ having certain properties). Roughly the fixed point of the system (still given by the smooth part of the dynamics) is $(h_0, h)$-stable if for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $h_0(t_0, x_0) < \delta$ implies $h(t, x(t)) < \varepsilon$, for any solution $x(t)$ of the system. By considering different sorts of functions $h_0$ and $h$, one can encompass various types of stability (In particular classical Lyapunov stability if $h$ and $h_0$ are the Euclidean norms). [316] gives sufficient conditions that guarantee stability (see e.g. corollary 3.6). Quite interestingly, [27] theorem 13.3 and [316] consider the situations in which $V_{rd}$ is only semi-negative definite and one wants to prove asymptotic stability, extending Matrosov’s result for nonautonomous continuous vectorfields.

Leela [307] and Rao [452] consider the impulses $Du$ as a perturbation of a Lyapunov stable smooth system with $u$ an exogeneous function in $RCLBV$, and study conditions on $Du$ (see hypothesis $(H_4)$ through $(H_6)$ in [307], definition 2.1 in [452]) such that stability is preserved (Roughly the jumps in $u$ must converge sufficiently fast to zero). Lyapunov functions are defined for measure systems, and can be used to prove asymptotic-self-invariance (ASI) [306] of the equilibrium point $x = 0$ of the smooth dynamics. (ASI is a stability concept adapted to perturbed systems with asymptotically vanishing disturbances: for instance the set $\{x = 0\}$ is ASI relative to $\dot{x} = -x + e^{-t}$ [306]).

The authors in [43] study controllability of systems of the form (7.1) with linear dynamics

$$\begin{cases} \dot{x} = Ax \\ \sigma_z(t_k) = B_k x(t_k^-) + C_k u_k \end{cases}$$ (7.3)

The simple bouncing ball system in (7.19) belongs to the class in (7.3) with $B_k \equiv 0$, $C_k^T = (0, \frac{1}{m})$, $u_k = p_k$. Then applying [43] theorem 2.3 one can study null-controllability of linear impacting systems, assuming this time that the percussions are exogeneous signals, so that no restitution law has to be considered (It is another problem to study how these percussions may be obtained via an additional system that produces some unilateral constraints). First the smooth part of the system (i.e. between impacts) must be stabilized and such that $[C_k, AC_k, \ldots, A^{n-1}C_k]$ has full rank $n$ for all $k$ (which lokks like a sort of mixed continuous-discrete controllability condition). Then from [43] there exists a sequence of controls $p_k \in l_q$ such that given any initial conditions, the state can be driven to zero in finite time (null-controllability). Notice that this results a priori deal with open-loop control in some admissible space $l_q$, and that [43] theorem 2.3 provides sufficient conditions only. It would be interesting to study state (i.e.pre-impact values) feedback, as well as extension to nonlinear systems. Finally let us remark that one nice feature of the results in [43] is that it does not rely on explicit integration of the system’s
7.1. GENERAL STABILITY CONCEPTS

dynamics to transform it into a discrete-time system, but uses mixed properties of the smooth and the nonsmooth parts. Also their result is similar to the ones in [27] [307] [452] in that the controlled system’s state converges to zero provided the impulses magnitudes are small in a certain sense.

Remark 7.2 Bainov and Simeonov [27] chapter 1 identify 3 classes of systems with impulse effect: the first class contain systems for which times $t_k$ are exogeneous; the second one concerns systems submitted to impulses with $t_k = t_k(x)$, i.e. the impact times are state-dependent; the third class fits within systems submitted to unilateral constraints. Quite interestingly, the work in [63] hinges on a similar classification to propose a general model of so-called hybrid dynamical systems $^2$, to which systems with unilateral constraints belong according to the definition in [63]. The authors also argue that autonomous switching phenomenon (i.e. when the vector-field of an ODE switches between different values, the conditions of switch depending only on the smooth system’s state, like for instance in [360]) can be viewed as a special case of autonomous jumps, i.e. systems with state unilateral constraints. This deserves further investigations since for instance control of manipulators during complete tasks are subject to both phenomena.

In view of the work in [63] one might wonder if mechanical systems subject to unilateral constraints could not be cast in a more general framework on nonsmooth dynamical systems, but: on one side it is clear that impact dynamics generate independent studies like restitution rules, and problems of motion of kinematic chains; on the other side the available general studies do not permit to conclude, because even simple impacting systems like the one-dimensional bouncing ball do not fit within the analysis in [27] [63] (see subsection 7.1.2). It is nevertheless worth realizing that discontinuous motion occupies such a large place in dynamical systems and control theory.

7.1.2 Stability of mechanical systems with unilateral constraints

The analytical tools described in subsection 7.1.1 on stability of certain MDE’s are in fact not well suited for impacting systems in general. This is due to several reasons that we try to explain now.

- Firstly, in many cases the smooth dynamics alone are not stable (or do not even possess any fixed point) and the impacts are really stabilizing the system. Consider for instance the simple example of a bouncing ball under gravity described below, where smooth dynamics are simply $m\ddot{x} = -g$, and merely adding $x \geq 0$, $\dot{x}(t^+_k) = -e\dot{x}(t^-_k)$, $x(t_k) = 0$ makes $x$ globally converge to the point $x = 0$ for $e \in [0, 1)$. In this case it seems quite difficult to have both conditions in (7.2) (see theorem 7.1) satisfied simultaneously.

\footnote{i.e. systems with both continuous dynamics and discrete phenomena.}
• Secondly finite accumulation points in \( \{t_k\} \) are not considered in [27] [307] [452] [316], see the conditions of theorem 7.1: this is quite annoying since for instance in a robotic task, the main goal may be to stabilize the robot’s tip in finite time on the constraint. The theory developed in [27] [316] is based on piecewise continuous solutions, and on the absence of the so-called beating phenomenon defined as follows: in the extended state space \((t, x)\), there is beating if a trajectory meets infinitely often the same hypersurface (For instance a surface in the configuration space of a mechanical system). This severely limits the applicability of the proposed theories since the one-dimensional rebounding ball problem with \( e \in [0, 1] \) and possible dissipation during flight times is not covered (The condition \( t_k - t_{k-1} \geq \delta > 0 \) is verified only if \( e = 1 \) and there is no dissipation on \((t_k, t_{k+1})\), or if \( e = 0 \). Such motions are called of finite sort in [379]).

• Thirdly and most importantly, it is supposed in theorem 7.1 that the fixed point of the system in (7.1) is given by the continuous time subsystem, i.e. \( f(t, 0) = 0 \). It is precisely the stability of this point that is analyzed. In the case of a mechanical system with a unilateral constraint, this corresponds to studying the stability of a point inside the domain \( \Phi \times \mathbb{R}^n \) defined by the constraint (i.e. \( \Phi \times \mathbb{R}^n = \{q, \dot{q} : f(q) \leq 0\} \)). Indeed, if the continuous-time dynamical equations possess a fixed point that is outside of \( \Phi \times \mathbb{R}^n \), then the fixed point of the overall system cannot be this point. It must then be given by combination of the smooth and the impulsive dynamics (i.e. the collision restitution law). Necessarily then, the equilibrium belongs to the boundary of \( \Phi \times \mathbb{R}^n \) denoted as \( \partial \Phi \times \mathbb{R}^n \) (in some robotic tasks we shall require to stabilize the robot’s tip at \( f(q) = 0 \) with \( \dot{q} = 0 \)). Note that all classical stability concepts rely on the notion of metric (hence topological) spaces [619], i.e. one assumes that it is possible to define neighborhoods (open domains) of the fixed point, and then one defines stability of this point. When this fixed point belongs to the boundary of the domain within which the state evolves (this domain is rendered invariant by the restitution rules), then such concepts are no longer applicable. It is possible that stability concepts for systems with the state constrained to evolve in a closed domain can be defined, see remark 7.4. We emphasize that the big difference between a mechanical system subject to a unilateral constraint together with a restitution rule, and systems like in (7.1), is that in the first case, one cannot get rid of the unilaterality condition: indeed, one could think of considering the dynamical equations (smooth part and jump conditions) and forget about the initial conditions compatibility with the constraint. But then a simple system that is stable when the unilaterality is considered (think of the bouncing ball example in (7.19)) would become unstable (indeed, if one assumes that the ball initial position is "under" the ground, then gravity makes it diverge from the origin, and the impact conditions become worthless). It is therefore necessary to keep the unilaterality (or half-space) feature of the problem. A possible path

\[3\]In a sense possibly different from that defined in definition 7.1.
to follow could be to consider the symmetric system by changing the sign of the smooth vector field when the position of the fictitious system is outside the constraint. We could then obtain two systems subject to unilateral constraints, and the concatenation of them would define a system whose state evolves in the whole state space. Note however that the first item has to be verified to comply with definition 7.1.

Notice finally that the Zhuravlev-Ivanov nonsmooth coordinate change (see subsection 1.4.2) allows to conclude on the stability of the equilibrium point of the transformed system. In a sense it allows to overcome this problem, but is limited to codimension one constraints (or codimension \(\geq 2\) but orthogonal constraints). See also remarks 7.14 and 7.4 below.

It must then be concluded that the framework in [27] applies well to the case when one wants to stabilize a manipulator close to an obstacle. Then some collisions may occur, and such criterion can be used to guarantee that these collisions will not make the state diverge. As we show in example 7.1, it is possible to design controllers such that both conditions in theorem 7.1 are fulfilled. Also piecewise continuous solutions and Lyapunov functions can be extended to \(RCLBV\) functions in a natural way, since \(RCLBV\) functions have a countable set of discontinuity points (see appendix C). Hence our first two objections are not really serious obstacles. The third one requires some investigations, and is the subject of subsection 7.1.4 below, and of chapter 8 where other stability concepts are presented, see also remark 7.7.

**Remark 7.3** Gopalsamy [177] considers (apparently this is the first work on this topic) stability of linear measure differential equations with delays. The above comments are still true, and this precludes direct application of the results to impacting systems. Extensions of this work are therefore needed. It might be interesting to study stability of the transition phase in robotics when for instance the velocity measurements are time-delayed: what is the effect of a delayed-damping on the system's behaviour? Assume a 1-degree-of-freedom robot is controlled with a PD input \(u = -\lambda_1 \dot{x}(t - \tau) - \lambda_2 x\) with measurement delay \(\tau\) on the velocity such that the closed-loop equation is

\[
\dot{m} \ddot{x} + \lambda_1 \dot{x}(t - \tau) + \lambda_2 x = 0, \quad x \leq 0
\]

(7.4)

where the functional dynamical equation is true during free motion phases, and we omitted the percussion rule. The dynamical equations are thus retarded measure differential equations (RMDE). The first problem to be solved is to assure the existence of the impacts sequence \(\{t_k\}\). Assume for instance that the initial conditions on \((-\tau, 0)\) (Recall that such functional differential equations require the definition of initial conditions on a whole time interval) are such that \(\dot{x} < 0\) on that interval. Then on \((-\tau, 0)\), \(u > 0\) so that the applied force on the mass has the right sign for stabilization on the surface \(x = 0\). Now for any \(\tau > 0\), \(\lambda_1 > 0\), \(\lambda_2 > 0\), there exists \(e \in [0, 1]\) and initial conditions such that the mass is stabilized at \(x(t_f) = \dot{x}(t_f) = 0\), with \(0 < t_f < \tau\). This lasts until \(t = \tau\). Then on \([\tau, 2\tau]\), \(u\) may take negative
values, so that the mass may detach from the surface \( x = 0 \). This constitutes a big difference with the case \( \tau = 0 \). Conditions have to be found such that this does not happen, or at least such that stabilization can be attained asymptotically. Note that the impacts are not memorized by the system, because of integration that produces a "smoothing" effect on the solution. Since the stabilization on the surface implies an infinite sequence \( \{ t_k \} \), the solution will have to be considered in \( RCLBV \). Note that to the best of our knowledge no study on retarded systems has considered initial conditions of bounded variation. Stabilization on obstacles via rigid bodies theory renders this mathematical setting necessary. In [398] only force feedback with time-delay has been considered, and no such problems arise since no RMDE appears in the closed-loop system.

Example 7.1 (Stabilization of a manipulator close to an obstacle) Let us consider the dynamics of a rigid manipulator as in (6.6). Let us however assume that the vectors \( n_q \) and \( t_{q,i}, 1 \leq i \leq n - 1 \), have not been normalized (in the sense of the kinetic metric). In other words, one constructs the basis with \( n_q = M^{-1}(q) \nabla_q f(q) \), and the vectors \( t_{q,i} \) are chosen mutually orthogonal and orthogonal to \( n_q \), i.e. verifying \( t_{q,i}^T \nabla_q f(q) = 0 \). It is then obvious that there exists \( r_n(t) \) such that \( \dot{n} = \dot{q} \), since we can simply choose \( r_n(t) = f(q(t)) \). Let us denote \( r_n \) as \( q_{\text{norm}} \) for notational coherence. Mimicking [607], we assume the existence of a vector function \( q_t(t) \) such that \( \dot{q}_t = h_{\text{tang}} \), and we denote \( r_t \) as \( q_{\text{tang}} \). Note that it may not be always possible to find out vectors \( t_q \) such that \( h_{\text{tang}} \) is integrable. Consider the lamina example treated in section 6.3.1. From (6.35) it is clear that except if \( H(\theta) \) is constant, then \( h_{\text{tang}} \) is not integrable. Let us write the dynamical equations in (6.6) as

\[
\begin{cases}
\dot{q}_{\text{norm}} + f_{\text{norm}}(q, \dot{q}) = n_q^T u + n_q^T p_q \\
\dot{q}_{\text{tang}} + f_{\text{tang}}(q, \dot{q}) = t_q^T u
\end{cases}
\tag{7.5}
\]

Define the variables \( s_{\text{norm}} = \dot{q}_{\text{norm}} + \lambda \dot{q}_{\text{norm}} \) and \( s_{\text{tang}} = \dot{q}_{\text{tang}} + \lambda \dot{q}_{\text{tang}} \). \( \tilde{q} = q - q_d \), where \( q_d \) is some (assumed here constant) desired value to be tracked. Then the following control input

\[
\begin{bmatrix}
{n_q^T} \\
{t_q^T}
\end{bmatrix}
= 
\begin{bmatrix}
{f_{\text{norm}}(q, \dot{q})} \\
{f_{\text{tang}}(q, \dot{q})}
\end{bmatrix} + 
\begin{bmatrix}
{-\lambda \dot{q}_{\text{norm}} - s_{\text{norm}}} \\
{-\lambda \dot{q}_{\text{tang}} - s_{\text{tang}}}
\end{bmatrix}
\tag{7.6}
\]

implies the following closed-loop equation during free-motion phases

\[
\begin{cases}
\dot{s}_{\text{norm}} + s_{\text{norm}} = 0 \\
\dot{s}_{\text{tang}} + s_{\text{tang}} = 0
\end{cases}
\tag{7.7}
\]

Note that provided \( q_{\text{norm,d}} < 0 \), the fixed point for the closed-loop equations in (7.7) is in the free-motion space. Let us consider the following positive definite function

\[
2V = s_{\text{norm}}^2 + \lambda \dot{q}_{\text{norm}}^2 + s_{\text{tang}}^2 + \lambda \dot{q}_{\text{tang}}^2 \tilde{q}_{\text{tang}}
\tag{7.8}
\]
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Then it is straightforward to verify that along the free-motion trajectories in (7.7) we get
\[ \dot{V} = -q_{\text{norm}}^2 - \lambda^2 \dot{q}_{\text{norm}}^2 - \dot{q}_{\text{tang}}^T \ddot{q}_{\text{tang}} - \lambda^2 \ddot{q}_{\text{tang}}^T \dot{q}_{\text{tang}} \leq 0 \]  
(7.9)

Now notice that at the impact times we have \( s_{\text{norm}}(t_k^+) = \dot{q}_{\text{norm}}(t_k^+) - \lambda q_{\text{norm},d} \), and similarly for \( t_k^- \). At the collision times we get
\[ \sigma V(t_k) = (e^2 - 1) \left( q_{\text{norm}}(t_k^-) \right)^2 + 2\lambda^2 q_{\text{norm},d}(e + 1) \dot{q}_{\text{norm}}(t_k^-) \]  
(7.10)

Now note that if the constraint is of the form \( f(q) = q_{\text{norm}} \leq 0 \), then the preimpact velocity must verify \( \dot{q}_{\text{norm}}(t_k^-) > 0 \). Indeed it must point outwards the constraint. Then one sees from (7.10) that \( \sigma V(t_k) \leq 0 \). Notice that if we had chosen the desired normal value inside the constraint, i.e. \( q_{\text{norm},d} > 0 \), this would have not been verified (besides of the above mentioned problem). Hence both conditions in theorem 7.1 are verified, together with the fact that the fixed point belongs to the smooth part of the system. It can thus be concluded that such a control law guarantees Lyapunov stability (in the sense of definition 7.1) of the equilibrium \( q = q_d, \dot{q} = 0 \), despite of possible collisions with the environment.

Remark 7.4 Ivanov [231] studies the extension of Lagrange-Dirichlet theorem (4) to \( n \)-degree-of-freedom systems with unilateral constraints. Lyapunov stability of equilibrium points \( q^* \) situated on the constraint surface and with zero reaction forces (i.e. just in contact) is investigated, in the lossless impacts case. Rapoport [453] develops a technique to check the conditions of [231] theorem 1. In fact it may be argued that despite that \( x^* \in \partial \Phi \times \mathbb{R}^n \), one just has to restrict the set of initial conditions to admissible ones. We however prefer to use techniques that allow to get rid of the unilaterality conditions, by either using the Zhuravlev-Ivanov transformation, or as we shall see the impact Poincaré map, and then speak of Lyapunov stability. This will be done in chapter 8 where a specific stability framework is developed.

7.1.3 Passivity of the collision mapping

In subsection 1.3.2 and in remark 7.1 we have highlighted the fact that systems with unilateral constraints can be interpreted as the concatenation of a smooth flow and a map. In subsections 2.1.1 and 2.1.3 we have seen that the work performed at the impact time by the impulsive contact forces is not well-defined if one chooses the Schwartz's distributions point of view, i.e. actually the point of view of solutions evolving as time functions. We have found out some approximating problems that have allowed us to assign some value to this work. It should be reiterated that this was possible because the very simple systems considered could be integrated. On the other hand, it is possible that several different approximating problems yield different results. In fact, the right approach is to consider the impact map and to study the dissipativity properties of this map. More details on dissipative systems

\[ ^4 \text{That relates the potential energy minimum and Lyapunov stability.} \]
can be found in appendix E. More precisely, consider the transformed dynamics in (6.6). Let us denote \( \hat{q} = \begin{pmatrix} \hat{q}_{\text{norm}} \\ \hat{q}_{\text{ang}} \end{pmatrix} \), and \( \hat{p}_q = \begin{pmatrix} n^T p_q \\ t_q^T p_q \end{pmatrix} \). Given a certain form of the percussion vector \( p_q \), one defines a restitution rule of the general form

\[
\hat{q}(t_k^+) = \hat{q}(k+1) = -\mathcal{E} \hat{q}(t_k^-) \triangleq -\mathcal{E} \hat{q}(k) \quad (7.11)
\]

\( \mathcal{E} \) is a matrix of restitution. We assume that \( \mathcal{E} \) is constant. Note however that in the following we look at the system only at an impact time \( t_k \), although the analysis treats the system as if it was an infinite sequence of iterations of a certain input/output map. In fact the restitution law at \( t_k \) is used to characterize an artificial discrete-time system, that is shown to be passive. Another possible path to follow would have been to study passivity properties from one impact to the next. But then we have to characterize the properties of the impact Poincaré map, and this is something completely different from what we do in this subsection.

In particular \( \mathcal{E} \) must be such that \( T_L \leq 0 \). From the kinetic energy loss form, this implies that \( \mathcal{E}^T \mathcal{E} \leq I_n \) (in the sense of inequalities between semi-positive definite matrices). We assume that the vectors \( n_q \) and \( t_q \) form an orthonormal basis with respect to the kinetic metric. At the collision time \( t_k \) there is a velocity jump and we can write

\[
\hat{q}(k+1) = \hat{q}(k) + \hat{p}_q(k) \quad (7.12)
\]

From (7.11) and (7.12) one deduces that

\[
\hat{q}(k) = -(I_n + \mathcal{E})^{-1} \hat{p}_q(k) \quad (7.13)
\]

Now consider the following output \( ^6 \)

\[
y(j) = \hat{q}(j) + \frac{1}{2} \hat{p}_q \quad (7.14)
\]

Notice that \( y(j) = \frac{1}{2} \hat{q}(j) + \frac{1}{2} \hat{q}(j+1) \). Consider now the system in (7.13) and (7.14). Defining the input as \( \hat{p}_q(j) \) and the output as \( y(j) \) we get

\[
\hat{p}_q(j)^T y(j) = \frac{1}{2} \hat{q}(j)^T \mathcal{E}^T \mathcal{E} \hat{q}(j) - \frac{1}{2} \hat{q}(j)^T \hat{q}(j) \quad (7.15)
\]

Hence it follows that the system in (7.13) and (7.14) defines a passive lossless system with storage function \( V(j) = \frac{1}{2} \hat{q}(j)^T \hat{q}(j) \). Under the condition that \( V(j+2) = V(j) \), i.e. \( T(t_j^+) = T(t_{j+1}) \), we can also write

\[
\sum_{j=0}^{n} \hat{p}_q(j)^T y(j) = V(n+1) - V(0) \geq -V(0) \quad (7.16)
\]

\(^5\)In chapter 6 we have already defined such matrices to describe multiple impacts, see for instance the rocking block example.

\(^6\)From now we use \( j \) instead of \( k \) to clearly separate the impact time \( t_k \) from the iterations of the artificial system we construct.
We have therefore defined a lossless system from the shock dynamical equations. However this is not quite interesting, because this does not reflect the energy loss property of the system at collisions. In fact the kinetic energy loss is absent from the considered system. We would rather like to obtain a strictly passive system associated to the impact map. This can be done as follows: consider the system

\[
\begin{align*}
x(j + 1) &= Ax(j) + Bu(j) \\
y(j) &= Cx(j) + Du(j)
\end{align*}
\] (7.17)

where \( u \in \mathbb{R}^n, y \in \mathbb{R}^n, x \in \mathbb{R}^n \). Then the following is true

**Claim 7.1** The system in (7.17) defines a strictly passive system with supply rate \( u(j)^T y(j) \), storage function \( V(x(j)) = \frac{1}{2} x(j)^T x(j) \) and dissipation function \( S(x(j)) = \frac{1}{2} x(j)^T [I_n - \mathcal{E}^T \mathcal{E}] x(j) \geq 0 \) provided the matrices \( A, B, C, D \) verify

- \( A^T A = \mathcal{E}^T \mathcal{E} \)
- \( B^{-T} C = A \)
- \( B^{-T} D^T B^{-1} = \frac{I_n}{2} \)

**Proof of claim 7.1**

Straightforward calculations yield

\[
u(j)^T y(j) = \frac{1}{2} x(j + 1)^T x(j + 1) - \frac{1}{2} x(j)^T x(j) + \frac{1}{2} [I_n - \mathcal{E}^T \mathcal{E}] x(j)
\] (7.18)

Note that we can choose \( A = \mathcal{E} \) as well as \( A = -\mathcal{E} \). From (7.18) we get that for \( u \equiv 0, V(j + 1) = V(j) - S(j) \leq V(j) \). Identifying \( x(j) \) with \( \dot{q}(j) \), the storage function is the kinetic energy of the system. The dissipation term \( S(j) \) represents the kinetic energy loss at impact. We have thus associated to the impact map a family of strictly passive systems whose storage function matches with the kinetic energy when the state is taken as the generalized transformed velocity \( \dot{q} \).

**Remark 7.5** The systems we have considered in (7.13) (7.14) and in (7.17) can be interpreted as follows: in the first case, the supply rate directly represents the kinetic energy loss \( T_L \). It can therefore be considered as the expression of the work of the impulsive forces at the impact time (see chapter 1 for comments on this point). The restitution rule has been used to define the system's equation, together with the shock dynamics. In the second case, the supply rate has no direct physical interpretation, since the input and output are defined more or less arbitrarily. But the system is strictly passive, i.e. the kinetic energy loss directly influences the system's state evolution. The restitution rule is included in the dissipation term \( S(x) \).

**Remark 7.6** If \( \mathcal{E} \) is the identity matrix \( I_n \), then the system is passive lossless. Then \( A = I_n \) and it can be verified that the conditions in claim 7.1 are equivalent to the KYP conditions in claim E.1.
CHAPTER 7. STABILITY OF SOLUTIONS OF IMPACTING SYSTEMS

7.1.4 Stability of the discrete dynamic equations

The second general way to characterize the stability of impacting systems is to study the discrete-time system associated to the overall dynamics. More precisely, by integrating the smooth vector field and incorporating the impact conditions, one is theoretically able to derive the impact Poincaré map of the system. As we shall see in this chapter and in chapter 8, the main problem is that such discrete-time systems cannot in general be obtained explicitly. But their stability and some of their properties can sometimes be studied without their closed-form.

The bouncing-ball with fixed obstacle example

To illustrate this in a simple case let us consider three other different ways of considering the simple bouncing ball example dynamics (recall the sweeping process formulation in (5.42)):

\[
\begin{align*}
\text{m} \ddot{x} &= -mg \\
x(t_k^+) &= -e \dot{x}(t_k^-) \\
x(t_k) &= 0 \\
\ddot{x}(t) &= 0 \text{ for } t > t_k \text{ if } \dot{x}(t_k^+) = 0
\end{align*}
\]

\[
\begin{align*}
\text{m} \ddot{x} &= -mg + \sum_{k=0}^{+\infty} p_k \delta_{t_k} \\
x(0) &= x_0 \geq 0 \\
m(\dot{x}(t_k^+) - \dot{x}(t_k^-)) &= p_k \\
\dot{x}(t_k^+) &= -e \dot{x}(t_k^-) \\
x(t_k) &= 0 \\
\ddot{x}(t) &= 0 \text{ for } t > t_k \text{ if } \dot{x}(t_k^+) = 0
\end{align*}
\]

\[
\begin{align*}
\text{m} \{ \dot{x} \} &= -mg \\
x(0) &= x_0 \geq 0 \\
\Delta_k &= \frac{2}{g} e^k \dot{x}(0) \\
\dot{x}(t_k^+) &= -e^{k+1} \dot{x}(0^-) \\
x(t_k) &= 0
\end{align*}
\]

where \( \Delta_k = t_{k+1} - t_k \). It is note worthy that the expression for \( \Delta_k \) is obtained by solving a second order algebraic equation, whose second solution is \( \Delta_k = 0 \). However, if \( \dot{x}(t_k) < 0 \), then it can be proved that there exists \( \delta > 0 \) such that on \( (t_k, t_k + \delta) \), \( x(t) > 0 \). Hence the zero solution must be disregarded. This manner of proceeding is generic when one wants to obtain the impact-discrete-time system associated to an impacting system. The main difference between such discretization and the classical discretization procedure for linear or nonlinear systems (see \[399\] chapter 14) is that for impacting systems, the "sampling" times (which are the instants of collision \( t_k \)) depend on the system's state, generally in a nonlinear way (which explains why convex systems as defined in chapter 1, section 1.3 are nonlinear). This renders those discrete systems very complex even if the separate dynamics are quite simple, and even hard to explicitly write down in general.

These three representations of the same system are equivalent since they exactly yield the same solution \( x(\cdot) \). The first one is a dynamical system with unilateral constraints, together with some impact law. The second one is a measure differential
system, and the third one is the discrete-time system associated with these dynamics. Notice that in the third formulation, if the restitution coefficient is zero, then the discrete time system yields that $\Delta_k = 0$, $x = 0$ and $\dot{x} = 0$ for $t \geq t_0^+$. Hence the condition that $\dot{x} = 0$ if $\dot{x}(t_k^+) = 0$ is not needed in this formulation. In the other two it is needed, otherwise for initial conditions $x(0) = 0$, $\dot{x}(0^+) = 0$ one gets $x(t) = -\frac{2m}{2m}t^2$ for $t \geq 0$, hence penetration into the constraint. On the contrary, imposing a zero acceleration in this case implies that the particle stays at rest on the constraint surface. Also we indicate that the initial position must be compatible with the constraint in the last two representations. Indeed the fact that the position must verify the unilateral constraint is not implied by the smooth and impact dynamical equations alone: it is verified if the initial condition is inside the domain $x \geq 0$. In other words this domain is invariant under these dynamical equations. We already noted such fact in the sweeping process formulation in section 5.3. As we discussed in the section 1.4.2 on the Zhuravlev-Ivanov nonsmooth change of coordinates, the condition is needed in the first two formulations to recall that when the system is at rest on the surface, the dynamical equations are algebraic. It is noteworthy that the transformations in (7.19) are obtained explicitly because the dynamics between impacts are very simple. The only addition of dissipativity during the flight-times renders the problem much more difficult. We discuss this point in chapter 8, subsection 8.5.2. In particular any small amount of dissipation $-\lambda \dot{q}$ precludes an explicit calculation of the flight-times and the finite time convergence of $\{t_k\}$ is much more difficult to be proved.

Remark 7.7 It is clear from (7.19) that the impact Poincaré map of the system is the natural way to study the stability of the fixed point $x = \dot{x} = 0$, and this is due to the fact that the fixed point of the continuous vector field does not exist. The same conclusion would hold if this fixed point existed, but would be outside $\Phi \times \mathbb{R}^n = \{x \geq 0\} \times \mathbb{R}$. The dynamical equations are quite different from those MDE's representing a continuous stable vector field perturbed by some impulses. When we deal with feedback control in chapter 8 we shall have to merge both concepts into a more general stability framework.

Lyapunov stability of discrete systems

The stability of systems with impulses can therefore be attacked via the discrete system (or impact map, or Poincaré impact map) associated to the total system. Notice that this map is different from the discrete map considered in the definition 7.1, which was simply the jump map of the system. Now we require the integration of the trajectories between the jumps to get the Poincaré impact map. Stability definitions and criteria for such maps are given in [164]. Let such a map be given by

$$x_{n+1} = f(x_n)$$

Assume that $f(x^*) = x^*$ for some fixed point $x^*$ of the map. Then we have
Definition 7.2 (Lyapunov discrete stability [164]) The fixed point $x^*$ of the map in (7.20) is stable in the sense of Lyapunov if and only if for all $n_0 \in \mathbb{N}$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $||x_0 - x^*|| < \delta$, then for all $n \geq n_0$, $||x_n(x_0, n_0) - x^*|| < \varepsilon$.

where $x_0$ denotes $x_{n_0}$, and $x_n(x_0, n_0)$ denotes the $n$-th iterated value starting with initial conditions $x_0$, $n_0$. Then Lyapunov second method provides the following result.

Theorem 7.2 ([164]) The fixed point $x^*$ of the map in (7.20) is Lyapunov stable if and only if there exist a continuous function $V(\cdot)$ and a class $K$ function $\alpha(\cdot)$, a ball $B_r(x^*)$ with radius $r$, centered at $x^*$, such that for all $x_n \in B_r(x^*)$

- $V(x_n) \geq \alpha(||x_n - x^*||)$
- $V(x_{n+1}) - V(x_n) \leq 0$
- $V(x^*) = 0$

Let us reiterate that both stability definitions in 7.2 and 7.1 are not equivalent. First of all, the stability in definition 7.1 concerns the whole system with state vector $x \in \mathbb{R}^n$. On the contrary, the second stability definition concerns a reduced order system (this is indeed the aim of Poincaré maps to reduce the dimension of the system by analyzing it only in one Poincaré section of the state-space. For the bouncing ball the section is $x = 0$). Furthermore, the bouncing ball example shows that in certain cases, the impact Poincaré map is easily derived, and Lyapunov stability as in theorem 7.2 can be proved. In this case it seems quite difficult to find out a Lyapunov function such that conditions like those of theorem 7.1 be satisfied. On the contrary, since it is not always obvious (often impossible, even for simple dynamics) to calculate the Poincaré map, a criterion like the one in theorem 7.1 may be more tractable to prove boundedness and convergence of the system's state towards some equilibrium point.

Note that the state variables for the discrete-time associated system may be chosen as $\dot{x}(t_k^*)$ and $\Delta_k$ [257]. The discrete mapping fixed point is given for the times $t_k$ as $t_\infty$ or as 0 for the flight-times $\Delta_{k+1}$. This is equivalent to studying the trajectories through the section $x = 0$ of the extended state space $(t, x, \dot{x})$. As we shall see in chapter 8, the choice of the flight time as a state variable looses its meaning when one considers a whole robotic task, where free-motion as well as constrained motion phases are merged. It is apparent from (7.19) that the bouncing ball problem includes a finite accumulation point of rebounds, i.e. the sequence $\{t_k\}$ is infinite and has a finite limit $t_\infty < +\infty$. This is really created by the discrete (or impulsive) dynamics themselves. It is a general fact that the impulsive dynamics may drastically modify the total dynamics. Let us consider as an additional proof of this fact the following:
Example 7.2 As another example, let us consider the following system \[ \begin{align*}
\dot{x} &= 1 + x^2, \quad x(0) = 0 \\
\sigma_x(t_k) &= -1, \quad t_k = \frac{k\pi}{4}, \quad k = 1, 2, \ldots
\end{align*} \tag{7.21} \]
Then the solution of the smooth dynamics, without any impulse, is given by \( x(t) = \tan(t) \), and therefore escapes in finite time. On the contrary, the solution of the whole system is given by
\[ x(t) = \tan\left(t - \frac{k\pi}{4}\right), \quad \frac{k\pi}{4} < t < \frac{(k+1)\pi}{4}, \quad k = 1, 2, \ldots \tag{7.22} \]
The solution in (7.22) is periodic with period \( \frac{\pi}{4} \), and jumps from 1 to 0 at \( t_k \). We have \( x(t_k^+) = 0 \) and \( x(t_{k+1}^-) = 1 \). By choosing either the post or the preimpulse values, it is easily visualized that the corresponding discrete map has a unique value at 0 or at 1. In the case of the bouncing ball, both post and preimpact maps possess a stable fixed point at \( x_k = 2^{k-1} = 0 \).

Hence the discrete dynamics have created a sort of limit cycle in the system (We say "a sort of" because the term limit cycle is in fact for second order systems, in the phase plane). Of course this example is first order, but it is remarkable that similar conclusions have been drawn for some simple mechanical systems (hence second order) by adding some impacting devices \[264\] \[265\].

Existence of periodic trajectories

A natural method to prove the existence of periodic trajectories in an impacting system is the following: assume that one seeks for trajectories with period \( T \) and two impacts per period. First calculate the solution of the system on \((t_k, t_{k+1})\) and on \((t_{k+1}, t_{k+2})\). Then search which conditions the system’s parameters have to satisfy so that \( \varphi(t_{k+1}^+, t_k, u_k) = \varphi(t_{k+2}^+, t_{k+1}, u_{k+1}) \) with \( u_k \) the solution at \( t_k^- \). This provides a set of initial data \( u_k \) and of parameters such that there exists such periodic trajectory \( T \). This has been employed in several studies, see e.g. \[347\] \[310\] \[483\] \[264\] \[265\].

Most importantly, notice that if the unilateral constraint is of codimension one, then the impacts necessarily occur with this surface, and between the impacts the system is free. But if there are several constraints, say 2, and if one looks for trajectories that collide one surface \( \Sigma_1 \) at \( t_k \) and then the other \( \Sigma_2 \) at \( t_{k+1} \) before a third impact with \( \Sigma_1 \) at \( t_{k+2} \), one must take care of the fact that the existence conditions have to incorporate that there is no impact with \( \Sigma_1 \) on \((t_k, t_{k+1})\), and not only the fact that there exists a strictly positive \( \Delta_{k+1} = t_{k+1} - t_k \) such that the trajectory attains \( \Sigma_1 \) in \((t_k, t_{k+1})\). In other words, certain existential results may include the possibility of trajectories which indeed attain \( \Sigma_2 \) after a strictly positive flight-time \( \Delta_{k+1} \), but which have to cross \( \Sigma_1 \) on \((t_k, t_{k+1})\). It seems that the works described in this subsection did not take into account such a constraint on the trajectory between the collisions.
Note that one implicitly chooses one constraint surface (there may be several) as the Poincaré section, and with velocities pointing inwards the admissible domain \( \Phi \). It is noteworthy that the found conditions are in general rather complex. Let us now describe in some detail the pioneering work in the field, by Masri and Caughey [347]. The system is depicted in figure 7.1, and consists of a ball with mass \( m \), sliding without friction on a block of mass \( M \). The ball is constrained by two limiting rigid stops. The absolute coordinates are denoted as \( y_a \) and \( x \) for the ball and the block respectively. The relative coordinate of the ball in a frame fixed with respect to the block is denoted as \( Y_r = y_a - x \). Hence \(-d \leq Y_r \leq d\) define the two unilateral constraints on \( Y_r \). We denote \( q = (x, y_a)^T \) the vector of generalized coordinates, and \( f_1(q) = -y_r + \frac{d}{2} \geq 0, f_2(q) = y_r + \frac{d}{2} \geq 0 \). Finally we employ the notation \( t^1_k \) for impacts with \( f_1(q) = 0 \), and \( t^2_k \) for impacts with \( f_2(q) = 0 \). The dynamical equations of the system can be written as:

\[
\begin{align*}
M \ddot{x} + c \dot{x} + kx &= A_0 \sin(\Omega t + \alpha) & \text{when } |y_r| > \frac{d}{2} \\
m \ddot{y}_a &= 0 & \text{when } |y_r| > \frac{d}{2} \\
\dot{q}(t^1_k)^T \nabla_q f_i(q) &= -\epsilon \dot{q}(t^1_k)^T \nabla_q f_i(q) & \text{where } f_i(q(t^1_k)) = 0, \dot{q}(t^1_k)^T \nabla_q f_i(q) < 0 \\
\end{align*}
\] (7.23)

It is a classical thing to calculate the block motion between impacts, that is given by:

\[
x(t) = \exp(-\delta \omega t) (B_1 \sin(\eta \omega t) + B_2 \cos(\eta \omega t)) + A \sin(\Omega t + \tau)
\] (7.24)

where \( B_1 \) and \( B_2 \) depend on initial data, and

\begin{itemize}
  \item \( \delta = \frac{c}{2\sqrt{kM}} \)
  \item \( \omega = \sqrt{kM} \)
  \item \( \eta = \sqrt{1 - \delta^2} \)
  \item \( \tau = \frac{\Omega}{\omega} \)
  \item \( A = \frac{A_0}{k \sqrt{(1 - \tau^2)^2 + 4\delta^2 \tau^2}} \)
  \item \( \tau = \alpha - \Psi \)
\end{itemize}
7.1. GENERAL STABILITY CONCEPTS

- \( \tan(\Psi) = \frac{2\pi}{1 - r^2}, \ 0 < \Psi < \pi \)

The goal is to find out conditions on the system's parameters (physical parameters, initial data, external excitation) such that there exists a trajectory with two impacts per period, i.e. \( t_{k+1}^i = t_k^i + \frac{\pi}{\Omega}, \ i, j = 1, 2, i \neq j \). Such a trajectory will exist if one is able to show the existence of \( B_1, B_2 \) and \( \tau \) which determine the block motion. The analysis proceeds first by fixing the following objectives:

\[
\begin{align*}
\dot{y}_a(t_k^{1+}) &= -v \\
\dot{y}_a(t_k^{1-}) &= v \\
\dot{y}_a(t_k^{2-}) &= -v \\
\dot{y}_a(t) &= (d + 2x(t_k^i)) \frac{\Omega}{\pi} = v & \text{for } t_k^i < t < t_k^i
\end{align*}
\]

(7.25)

In fact the third condition comes from the assumption that the ball evolves freely between collisions, with velocity magnitude \( v \), and has to slide on a distance equal to \( d + 2x(t_k^i) \) (in the absolute coordinate frame). Those conditions impose some restrictions on the type of periodic trajectories one is looking for (8). Indeed not only one fixes the number of impacts per period, but also the pre and postimpact absolute velocities of the ball (9). Using the shock dynamical equations, one can compute the postimpact velocities of the ball and of the block. Then using the conditions in (7.25), it is possible to derive expressions relating \( x(t_k^i) \) to \( x(t_k^{1+}) \) or to \( x(t_k^{2-}) \), as:

\[
\begin{align*}
x(t_k^i) &= -\frac{\pi}{2\Omega} \left( \frac{1+\varepsilon}{1-\varepsilon+2\zeta} \right) x(t_k^{1-}) - \frac{d}{2} \\
x(t_k^i) &= -\frac{\pi}{2\Omega} \left( \frac{1+\varepsilon}{1-\varepsilon+2\zeta} \right) x(t_k^{1-}) - \frac{d}{2}
\end{align*}
\]

(7.26)

where \( \varepsilon = \frac{\eta^2}{\Omega^2} \). The next step is to use the block equation of motion obtained from (7.23): if one replaces the block position and velocity in (7.26) by the calculated ones from the first equation in (7.23), one finds a complex set of equations which relate all the system's parameters with the clearance size \( d \), of the form:

\[ \mathcal{P}(\delta, \omega, \Omega, r, \tau, \eta, \kappa, \varepsilon)\zeta = \ddot{d} \]

(7.27)

where \( \zeta^T = (x(t_k^i), \dot{x}(t_k^{1-}), \dot{x}(t_k^{1+}), B_1, B_2, A) \) and \( \mathcal{P}^T = (0, 0, 0, 0, -\frac{d}{2}, -\frac{d}{2}) \). \( \mathcal{P} \) is a \( 6 \times 6 \) matrix. The aim is to determine whether this system admits a solution. For the sake of briefness, we do not recall here the explicit form of the matrix \( \mathcal{P} \) (see Masri and Caughey [347] for details). We simply mention that this system of

---

\(^8\) Notice that it is not possible in this case to have a periodic trajectory with collisions occurring repeatedly on the same constraint, since the ball is horizontally free between impacts. This however is possible with other systems, like the inverted pendulum in a box, see Shaw and Rand below.

\(^9\) In relationship with the fact that the calculated jump in the velocity, is independent of the fact that the used frame is Galilean or not (see chapter 4, subsection 4.1.3), it is clear here that the frame fixed with respect to the block is not Galilean, and does not satisfy the smoothness requirements discussed in subsection 4.1.3 equation (4.26), since \( \dot{z} \) is discontinuous at impacts. In other words, the jump of the absolute velocity \( \dot{y}_a \) is clearly different from that of the relative velocity \( \dot{y}_r \).
algebraic equations can be solved, and one obtains:

\[
\begin{align*}
A &= \frac{d}{2s} [h_1(\sigma_1 \theta_2) - (\sigma_1 \theta_1 + \eta \sigma_2 \omega)(1 + h_2)] \\
B_1 &= \frac{d}{2}(1 + h_2)(\sigma_2 - \sigma_1)C \\
B_2 &= \frac{d}{2}h_1(\sigma_1 - \sigma_2)C
\end{align*}
\]  
(7.28)

with

\[
\Delta = h_1 [C(\sigma_1 - \sigma_2) - (S + C \sigma_2)\sigma_1 \theta_1 + (S + C \sigma_1)\delta \omega \sigma_2] \\
+ (1 + h_2) [(S + C \sigma_2)\sigma_1 \theta_1 + (S + C \sigma_1 - \eta \omega \sigma_2]
\]

\[
S = \sin(\tau), \quad C = \Omega \cos(\tau),
\]

\[
h_1 = \exp\left(-\frac{\delta \pi}{\tau}\right) \sin\left(\frac{\pi}{\tau}\right)
\]

\[
h_2 = \exp\left(-\frac{\delta \pi}{\tau}\right) \cos\left(\frac{\pi}{\tau}\right)
\]  
(7.29)

\[
\sigma_1 = \frac{\pi}{2 \Omega} \frac{1+e}{1-e+2e} \\
\sigma_2 = \frac{\pi}{2 \Omega} \frac{1+e}{1-e-2e}
\]

\[
\theta_1 = \omega \exp\left(-\frac{\delta \pi}{\tau}\right) \left[-\delta \sin\left(\frac{\pi}{\tau}\right) + \eta \cos\left(\frac{\pi}{\tau}\right)\right]
\]

\[
\theta_2 = \omega \exp\left(-\frac{\delta \pi}{\tau}\right) \left[-\delta \cos\left(\frac{\pi}{\tau}\right) - \eta \sin\left(\frac{\pi}{\tau}\right)\right]
\]

It can be seen that the last two equations in (7.28) directly define the set of initial data which are needed to obtain a periodic trajectory with two impacts per period. The first equation in (7.28) is put in \([347]\) under the form

\[
2 \sin(\tau) + H \cos(\tau) = -\frac{d}{A}
\]  
(7.30)

for some \(H(\sigma_1, \sigma_2, \delta, \omega, h_1, h_2, \theta_1, \theta_2, \Omega)\). For the searched trajectories to exist, one must be able to compute \(\sin(\tau)\) and \(\cos(\tau)\). This implies conditions on the coefficients of the equation in (7.30). In particular one finds the condition \(\left(\frac{d}{A}\right)^2 \leq H^2 + 4\), which means that the clearance \(d\) should not be too large. Then the motion of the block is entirely known (since \(B_1, B_2, A\) and \(\tau\) are known), and that of the ball also. It is therefore shown that 2 periodic symmetric trajectories with 2 impacts per period and period \(\frac{2\pi}{\Omega}\) exist when the parameters satisfy the suitable conditions.

The result of Masri and Caughey allows one to determine analytic (complex in their form) conditions such that a certain type of periodic trajectories exists in the system. It is noteworthy that one can employ a similar path to investigate other types of typical motions. For instance, let us also reproduce a result of Shaw
and Rand [483], who analyze a quite similar system as Masri and Caughey. The dynamical equations are those of an inverted pendulum in a rigid box, see figure 7.2. The top of the pendulum is striking both vertical surfaces of the box (an impact oscillator). The box (base) is sinusoidally excited. The non-dimensional equations (with the approximation \(\sin(q) = q\)) are given by:

\[
\begin{align*}
\frac{\dot{q}}{} + 2\alpha q - q &= \beta \cos(\omega t) & \text{if } |q| > 1 \\
\dot{q}(t_k^+) &= -e\dot{q}(t_k^-) & \text{if } |q| = 1
\end{align*}
\] (7.31)

The second equation models the impacts of the pendulum with the faces of the box. The Poincaré section is chosen as above, i.e. \(\Sigma = \{q, \dot{q}, \eta : q(t_k) = 1, \dot{q} = \dot{q}(t_k^+) > 0\}\), with \(\eta = t \mod \text{the period } T = \frac{2\pi}{\omega}\) of the external excitation. \(\Sigma\) is therefore a section of codimension 1 in the extended 3-dimensional state space of the pendulum, and the postimpact velocity is chosen to define the Poincaré impact map. Let us denote \(\dot{q}(t_k^+) \equiv \dot{q}_k\) and \(\eta_k\) the \(k^{th}\) iteration of the Poincaré map \(P_\Sigma\). There are many different possible typical trajectories: some where the pendulum remains stuck on one of the walls (\(|q| = 1\)), some where it collides only one side, or both, periodic ones with \(m\) impacts per period, chaotic motions, \ldots. The following is true:

**Theorem 7.3 (Shaw and Rand [483])** A point \((q_k, \eta_k) \in \Sigma\) is a fixed point of \(P_\Sigma\) and corresponds to a subharmonic trajectory with period \(nT\) and one impact per period (i.e. the pendulum does not touch the walls on \((t_k, t_k + nT))\) if the following conditions are satisfied:

\[
\begin{align*}
[I_k^2 + \omega^2 G_k^2] \dot{q}_k^2 - 2G_kH_k\omega^2 q_k + \omega^2 H_k^2 \left[1 - \frac{\theta^2}{\zeta^2}\right] &= 0 \\
\sin(\omega \eta_k + \Psi) &= -\frac{\dot{q}_k l_k^2}{\beta \omega H_k} \\
\cos(\omega \eta_k + \Psi) &= -\frac{(H_k - \dot{q}_k G_k) \zeta}{\beta H_k}
\end{align*}
\] (7.32)
where
\[
\begin{align*}
H_k &= \cosh(kT\Omega) - \cosh(kT\alpha) \\
G_k &= \frac{1+\varepsilon}{2\varepsilon} \sinh(kT\Omega) \\
I_k &= \frac{1-\varepsilon}{2} H_k + \left[ \frac{1+\varepsilon}{2} \right] \left[ \frac{\varepsilon}{6} \sinh(kT\Omega) - \sinh(kT\alpha) \right] \\
\Omega &= \left[ 1 + \varepsilon^2 \right]^{\frac{1}{2}} \\
\zeta &= [(1 + \omega^2)^2 + (2\omega)^2]^2 \\
\tan(\Psi) &= \frac{2\omega}{1 + \omega^2}
\end{align*}
\]

One sees that the first equation in (7.32) is an algebraic second order equation in \(\dot{q}_k\). It is thus easy to derive conditions such that this equation admits zero, one or two strictly positive real roots. Conditions on the external excitation magnitude \(\beta\) are derived in [483] for saddle-node bifurcations after which the subharmonic trajectories described in theorem 7.3 appear. Then the authors investigate the stability of periodic trajectories using a technique described in the next subsubsection.

**Further comments** We have reproduced in some detail the results in [347] and [483] to illustrate once again the gap between the simplicity of the reasoning used to search for periodic trajectories, the (apparent) simplicity of the dynamical equations, and the complexity of the calculations. The system as in [347] that consists of a loose auxiliary mass \(m\) which impacts against the ends of a container fixed to the primary mass \(M\) is called an impact damper. When there are several loose masses \(m_i\), this is a multi-unit impact damper. Due to collisions an attenuation of the amplitude of vibration of the principal mass \(M\) may be achieved within a certain range of the forcing term frequency. The potential applications of such devices are to reduce vibrations in switching relays, turbine buckets, antennas, lathe tools, airplane ailerons, helicopter tension rods ..., and also the study of dynamics of systems with clearances. Other references on the impact damper are [146] (numerical study to show resonance behaviour) [150] (conditions of existence and stability of periodic trajectories with 2 impacts per period, where the impacts are represented by instantaneous coupling between the two masses, corresponding to plastic impact as in [264] [265]) [191] (basically use the Poincaré map Jacobian \(DP_x\) to study the stability of 2 impacts per period motions, and derives numerically a bifurcation analysis) [31] [441] (derive closed-form solutions as Masri and Caughey, but for other kinds of periodic trajectories, and present numerical and experimental results) and also [346] [97] [179] [576] [467] [138] [162] [163] (study of the dynamics of a percussive rock-drilling machine) [332] (multi-unit dampers with Coulomb friction effects between the loose mass and the principal mass) [466] [499] (application to printing machines). Other results on the existence of periodic trajectories can be found in [292]. They concern a particular type of billiards, which consist of a
free particle moving in a closed bounded domain of the plane. The collisions are assumed to be elastic, hence the flow with collisions is conservative \(^{(10)}\). Then some extensions of Poincaré's results on the existence of periodic trajectories for continuous systems (like the well-known Poincaré-Bendixson lemma \([554]\)) can be used for such impacting systems. See \([292]\) chapter 2 for more details. In this setting, Birkhoff has proved the following:

**Theorem 7.4** ([494]) Inside a bounded convex domain \(\Phi \subset \mathbb{R}^n\) with \(\partial \Phi\) smooth, there always exist infinitely many periodic trajectories.

Those periodic trajectories are polygons in \(\Phi\), according to definition 7.3. A refined formulation of Birkhoff's theorem is given in \([292]\) chapter 2, theorem 1, which takes into account the number of impacts per period.

Also in relationship with the Zhuravlev-Ivanov nonsmooth transformation (see chapter 1), one may use the theoretical background attached to differential equations with discontinuous right-hand-side (hence differential inclusions, see remark 1.15) to prove existence of periodic trajectories. See \([154]\) chapter 3, and the bibliography listed at the end of §14 of this reference. Let us mention also other works on existence of various types of periodic trajectories (more complex than those above), see e.g. Peterka's studies in \([431]\) \([432]\) \([433]\) \([288]\) \([434]\) \([289]\).

**Further comments on the Poincaré impact map stability analysis**

The global transformation of the dynamics into recurrence equations relies on strong properties of the trajectories (e.g. boundedness, periodicity) and on the ability of explicitly obtaining the solutions between impacts. This is a hard task except in very simple cases. In slightly more complex cases, the recurrence equations may still be obtained in an implicit form (because the flight-times durations cannot be obtained explicitly), see e.g. \([574]\) \(^{(11)}\). This may be illustrated as follows: we have seen in chapter 1 that a system with unilateral constraint can be considered as a

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\(^{(10)}\) In fact, Sinai \([494]\) defines a billiard as follows:

**Definition 7.3 (Billiard [494])** Given a bounded region \(\Phi \subset \mathbb{R}^n\), \(n \geq 2\), with \(\partial \Phi\) piece-wise continuous, a billiard in \(\Phi\) is a dynamical system generated by the uniform linear motion of a material point inside \(\Phi\), with a constant velocity and with reflection on \(\partial \Phi\) such that the tangential component of the velocity remains constant and the normal component changes sign.

The cases when the particle strikes a corner of \(\partial \Phi\) are neglected, because they belong to a zero-measure set \([494]\) and have no influence on the properties of the orbits. Note however that the existential results in \([417]\) allow to assert that there is a solution to such a dynamical problem, due to the conservation of energy assumption. Clearly although some mechanical systems may be recast into this definition, the application to mechanical engineering problems of billiards theory remains quite weak.

\(^{(11)}\) For instance equation (16) in \([210]\) are just a particular case of the general discrete-time system in \([574]\).
flow with collisions, see section 1.3:

$$\varphi_t : \mathbb{R}^{2n} \to \partial \Phi \times \{-V(q(t_0))\} \to \partial \Phi \times V(q(t_0)) \to \partial \times \{-V(q(t))\} \to \ldots$$

$$\cdots \to \partial \Phi \times \{-V(q(t_k))\} \to \partial \Phi \times V(q(t_k)) \to \mathbb{R}^{2n}$$

$$u_0 \mapsto \varphi(t_0^-; 0, u_0) \to \varphi(t_0^+; 0, u_0) \mapsto \varphi(t_1^-; 0, u_0) \to \varphi(t_1^+; 0, u_0) \to \ldots \mapsto \varphi(t_k^-; 0, u_0) \to \varphi(t_k^+; 0, u_0)$$

$$\cdots \mapsto \varphi(t_k^-; 0, u_0) \mapsto \varphi(t; 0, u_0)$$

(7.34)

To go from the flow with collisions $\varphi_t$ to the impact Poincaré map $P_\Sigma$ one may consider the following steps (12):

- Let us denote:

$$P : \partial \Phi \times V(q(t_k)) \to \partial \Phi \times V(q(t_{k+1}))$$

$$u(t^+_{k+1}) = P(u(t_k^+))$$

(7.35)

the mapping such that $P(u(t_k^+)) = \varphi_{t_k^+}(u_0)$, $u_0$ the initial data (notice that $u_0$ needs not to belong to $\partial \Phi \times V(q)$. This equality merely means that the value of $P$ at $t_k^+$ is given by the value of the flow with collisions with an admissible initial data $u_0$, considered at $t_k^+$. $P$ may be called the section map of the flow $\varphi_t$. Clearly $u(t^+_{k+1}) = P(u(t_k^+)) = F_{k+1}0\varphi_{t^+_{k+1}-t_0}(u_0) = F_{k+1}0\varphi_{t^+_{k+1}-t_0}(u(t_k^+))$, where $t_0$ is the initial time, i.e. $u_0 = u(t_0)$. If we want to consider the Poincaré map from $t_k$ to $t_{k+2}$ (2 impacts per period for instance, that could be named the second return map instead of the first return map) then we have to compute $u(t^+_{k+2}) = P(u(t_{k+1}^+)) = F_{k+2}0\varphi_{t^+_{k+2}-t_{k+2}}(F_{k+1}0\varphi_{t^+_{k+2}-t_{k+1}^+}(u(t_k^+))).$ Then by the chain rule one gets:

$$DP(u(t_k^+)) = D F_{k+1}0D \varphi_{t^+_{k+2}-t_{k+1}^+}0D F_{k+2}0D \varphi_{t^+_{k+2}-t_{k+1}^+}(u(t_k^+))$$

(7.36)

(Note that $DP(u_0)$ denotes the linear differential operator of $P$ calculated at $u_0$, such that $DP(u_0) = \frac{\partial P(u_0)}{\partial u_0}$). Notice that $F_k$ may not be constant but may in general depend on $q$, see the examples in (6.37) through (6.39).

- Now to compute the impact Poincaré map $P_\Sigma$ one has to first clarify the definition of its state vector that we denote as $\tilde{u}_\Sigma$ (13). The section is $\Sigma = \{u : f(q) = 0\}$ (with a codimension one constraint, i.e. $f(q) \in \mathbb{R}$). The most natural way to proceed is to introduce the quasi-coordinate $\tilde{q}_1 = f(q)$ and to

\footnote{12We shall need the following developments in chapter 8, lemma 8.2 when we deal with closed-loop stability.}

\footnote{13The reason for this apparently complicated notation is that we shall need several steps to go from $u$ to $\tilde{u}_\Sigma$.}
7.1. GENERAL STABILITY CONCEPTS

assume that the transformation:

\[ G : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \]
\[ u \mapsto \tilde{u}^T = (\tilde{q}_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n) \]  \hspace{1cm} (7.37)

is a global diffeomorphism \(^{14}\). Hence \( \tilde{u} = G(u) \) and \( u = G^{-1}(\tilde{u}). \) Therefore \( P(u(t^+_k)) = PoG^{-1}(\tilde{u}(t^+_k)) \). Now notice that

\[ \tilde{u}(t^+_k)^T = (0, q_2, \ldots, q_n, \dot{q}_1(t^+_k), \dot{q}_2(t^+_k), \ldots, \dot{q}_n(t^+_k)) \]  \hspace{1cm} (7.38)

and define

\[ \tilde{u}_{\Sigma, k}^T = (q_2(t_k), \ldots, q_n(t_k), \dot{q}_1(t^+_k), \dot{q}_2(t^+_k), \ldots, \dot{q}_n(t^+_k)) \]  \hspace{1cm} (7.39)

Then the impact Poincaré map value at \( \tilde{u}_{\Sigma, k} \) is given by:

\[ P_{\Sigma}(\tilde{u}_{\Sigma, k}) = PoG^{-1}(\tilde{u}(t^+_k)) \]  \hspace{1cm} (7.40)

i.e. \( P_{\Sigma} \) is the restriction of \( PoG^{-1} \) to \( \Sigma = \{ \tilde{u} : \dot{q}_1 = 0, q_1 > 0 \} = \partial \Phi \times V(q(t_k)) \), and \( P_{\Sigma} : \Sigma \to \Sigma \).

\textbullet Finally, although the value taken by \( P_{\Sigma} \) at \( \tilde{u}_{\Sigma, k} \) is given by (7.40), its explicit calculation requires to be able to express \( \tilde{u}_{\Sigma, k+1} \) as a function of \( \tilde{u}_{\Sigma, k}, \) i.e. \( \tilde{u}_{\Sigma, k+1} = P_{\Sigma}(\tilde{u}_{\Sigma, k}). \) This is in general impossible, because this hinges on the explicit calculation of the impact times \( t_k \), which cannot in general be explicitly obtained. However as we shall see below, in certain cases the Jacobian of \( P_{\Sigma} \) can be explicitly calculated.

Remark 7.8 A nice property of the section map is that contrarily to the flow with collisions, it does not depend explicitly on the collision times. Hence its Jacobian can be calculated. This follows from the fact that to obtain \( P \), one must be able to calculate the flight-times \( \Delta_{k+1} = t_{k+1} - t_k \), and that \( \Delta_{k+1} \) is a function of \( u(t^+_k) \). Also it clearly appears from the definition of the mapping \( G \), i.e. of the state vector \( \tilde{u} \), that the Jacobian of \( PoG^{-1} \) calculated around a periodic trajectory \( \tilde{u}^* \) always possesses an eigenvalue equal to 1. Indeed \( PoG^{-1} : \tilde{u}(t^+_k) \mapsto \tilde{u}(t^+_k) \) and one notices that the first component \( \dot{q}_1 \) of \( \tilde{u} \) is constant (zero) for all \( k \geq 0 \). Let us pose \( \tilde{u}(t^+_k) = \tilde{u}_k = \tilde{u}^* + \delta_k \), i.e. \( \delta_k \) is a small disturbance of \( \tilde{u}^* \). The Taylor expansion of \( PoG^{-1} \) gives \( PoG^{-1}(\tilde{u}_k) = PoG^{-1}(\tilde{u}^*) + D_{\tilde{u}}PoG^{-1}(\tilde{u}^*)\delta_k + O(\delta_k^2) = \tilde{u}^* + D_{\tilde{u}}PoG^{-1}(\tilde{u}^*)\delta_k + O(\delta_k^2) \).

Hence locally around \( \delta_k = 0 \) one gets the linear mapping \( \delta_{k+1} = D_{\tilde{u}}PoG^{-1}(\tilde{u}^*)\delta_k \). From the above remark one has \( \delta_{k+1} = \delta_{k,1} \) from which it follows that the first row of \( D_{\tilde{u}}PoG^{-1}(\tilde{u}^*) \) is equal to \( (1, 0, \ldots, 0) \) so that one eigenvalue of the Jacobian is always 1. Note that this is a general remark about Floquet’s matrices (see for instance [181] p.25) that the eigenvalues of \( D_{\tilde{u}}PoG^{-1}(\tilde{u}^*) \) (called the Floquet or 14In most of the cases it is clear that this will imply a reordering of the generalized coordinates. For instance if \( f(q) = q_2 \), then evidently one will not define \( G \) as above, but rather first exchange \( q_1 \) and \( q_2 \) in \( q \), or simply define \( G \) with \( q_1 \) as the second component of \( \tilde{u} \).
characteristic multipliers) are such that the one associated with perturbations along the periodic trajectory is unity. When considering the Poincaré map this eigenvalue disappears so that the remaining ones characterize the stability of the trajectory in question.

**Remark 7.9** Let us recall a general feature about flows (which is not particular to flows with collisions, but has however been used in this setting in [586] for a stability analysis of an \( n \)-balls system). Let the free-motion vector field be \( G(u) \). Then

\[
D_u \varphi_i^j(G(u)) = G(\varphi_i^j(u))
\] (7.41)

It is assumed that these quantities are calculated outside possible discontinuities. The result follows from the fact that by definition \( \frac{d}{dt}(\varphi_i^j(u)) = G(\varphi_i^j(u)) \) between impacts. Hence \( D_u \varphi_i^j(G(u)) = D_u \varphi_i^j(\dot{u}) = \left( \frac{\partial \varphi_i^j}{\partial u} (u) \right)^T \dot{u} = \frac{d}{dt} [\varphi_i^j(u)] = G(\varphi_i^j(u)) \) where we also used the fact that the trajectory \( \varphi_i^j(u) \) emerges from \( u \), such that \( \dot{u} = G(u) \) (\( \dot{u}(t_k^+) = G(U(t_k)) \)). Consequently for all \( u \) such that \( u - u_0 \in \text{span}\{G(u)\} \), one has that \( D_u \varphi_i^j(u) = D_u \varphi_i^j(u_0 + v) \), with \( v = \frac{G(u)}{||G(u)||} \), hence \( D_u \varphi_i^j(u) = D_u \varphi_i^j(u_0) + G(\varphi_i^j(v)) \). One thus remarks that it is sufficient to define the linear differential operator \( D_u \varphi_i^j \) on the quotient space \( T_u \) of the tangent hyperplane \( T_u(\Phi \times V(q)) \) (15) by the one-dimensional subspace \( \text{span}(G(u)) \) (i.e. the space that consists of equivalent classes formed by the equivalence relation \( u \sim u_0 \) if \( u - u_0 \in \text{span}\{G(u)\} \)). Note that evidently \( \text{span}(G(u)) \subset T_u(\Phi \times V(q)) \). Hence instead of considering the linear operator \( D_u \varphi_i^j : T_u(\Phi \times V(q)) \rightarrow T_{\varphi_i^j}(\Phi \times V(q)) \), one may consider the quotient linear operator \( L_u : T_u \rightarrow T_{\varphi_i^j(u)} \). One can also choose to identify \( T_u(\Phi \times V(q)) \) with \( T_u \) so that \( D_u P = L_u^\Phi \) [586], where \( P \) is the section map of the flow \( \varphi_i^j \) defined above.

\[ \nabla \nabla \]

Thus in order to explicitly obtain the impact Poincaré map \( P_u \) (16), one must be able to calculate the impact times \( t_0, t_1, \ldots \). When this is not possible, those times can be obtained in an implicit form, see (1.35). In a more general setting, the impact section map and the impact Poincaré map are implicitly expressed from (1.34) as:

\[
\begin{align*}
fo \varphi_q(t_{k+1}; t_k, u_k) &= 0 \quad \text{with } u_k = u(t_k^+) \\
u_{k+1} &= I_{k+1}(u(t_{k-1}^+)) \\
u(t_{k-1}^-) &= u_k + f(t_k, t_{k+1}) G(u(t)) dt
\end{align*}
\] (7.42)

\[ \text{Recall that when } q \in \text{Int}(\Phi) \text{ then } V(q) = \mathbb{R}^n, \text{ see definition 5.1. Thus } T_u(\Phi \times V(q)) \text{ can be identified with } \mathbb{R}^{2n}, \text{ equipped with the basis } \left\{ \frac{\partial}{\partial u_i} \right\}_{1 \leq i \leq 2n} [399]. \]

\[ \text{Notice that rigorously, Poincaré maps are defined for systems that possess periodic closed trajectories [181] §1.5. The term is employed for impacting systems with a little abuse since the trajectories will not necessarily be periodic (think of the bouncing ball example).} \]
It is also possible to make the flight-times $\Delta_{k+1} = t_{k+1} - t_k$ explicitely appear in this formulation by simply replacing $t_{k+1}$ by $t_k + \Delta_{k+1}$ \(^{(17)}\). Now notice that $P_z$ can be expressed as $P_z = P_{r,k} \circ P_{f,k}$, where $P_{r,k}$ corresponds to the restitution mapping $\mathcal{F}_k$, whereas $P_{f,k}$ corresponds to the flow between impacts. For instance assume that $f(q) = q_1$ as above. Then $P_{f,k} : \tilde{u}_{E,k} \mapsto \tilde{u}_{E,k}(t_k^-)$ and $P_{r,k} : \tilde{u}_E(t_k^-) \mapsto \tilde{u}_{E,k+1}$. Clearly $P_{r,k}$ can be simply expressed from the restitution mapping $\mathcal{F}_k$. The problem is to calculate $t_{k+1}$ to get the explicit form of $P_{f,k}$. This may be done \textit{via} an implicit equation of the form

$$h(t_{k+1}, t_k, u_k) = 0 \quad (7.43)$$

similar to (1.35). Now notice that the Jacobian of $P_z$ at a point $\tilde{u}_{E,0}$ is given from the chain rule by \(^{(18)}\) $DP_z \triangleq \frac{\partial P_z}{\partial \tilde{u}_E}^T(\tilde{u}_{E,0}) = \frac{\partial P_{r,k}}{\partial \tilde{u}_E}^T(y) \frac{\partial P_{f,k}}{\partial \tilde{u}_E}^T(\tilde{u}_{E,0}) = DP_{r,k}(y)DP_{f,k}(\tilde{u}_{E,0})$, with $y = P_{f,k}(\tilde{u}_{E,0})$. Obviously, we have not made a big advance if these Jacobians are not known explicitly. It is noteworthy that in certain cases, $t_{k+1}$ may be known only implicitly, while $DP_z$ is known explicitly. For instance an example is treated in [369] where the equation in (7.43) cannot be solved to yield $t_{k+1}$ as a function of $t_k$ and $u_k$. But one is able to derive conditions on the system's parameters such that a periodic motion, with specified period $T$, exists (similarly as for the Masri and Caughey example above). Hence one is able to express $DP_z$ as a function of $T$ and $t_k$, $\tilde{u}_E$ and calculate its eigenvalues (to apply Floquet's theory for instance, to check the stability of the fixed point of $P$, or to investigate the type of bifurcation which occur when parameters are varied, see [227]). It happens that the bifurcation condition on the excitation magnitude is independent of $t_k$ [310] [483]. The eigenvalues of the Floquet's matrix can thus be investigated in function of the system's parameters (input magnitude and period, dissipation coefficient, restitution coefficient), and the type of bifurcation can locally be deduced [227] chapter 7.

**Other studies on stability** Wang [574] deals with a lamina submitted to a time-varying unilateral constraint, in relation with catching tasks in robotics. The flight-times are given implicitly only in general, from the first equation in (7.42). He linearizes $P_z$ and assumes that conclusions about the linearized map can be carried to the nonlinear system in the degenerate case when there is a continuum of fixed points. It would be interesting to analyze this impacting system \textit{via} the Zhuravlev-Ivanov transformation (see chapter 1, subsection 1.4.2). In particular local stability investigations are possible for the transformed system, see [235]. The global analysis that take into account the nonlinearities effects will generally require a numerical procedure [573]. Many other studies contain a stability analysis of periodic trajectories, see e.g. [150] (stability analysis of an impact damper) [339] [340] [431] equation (4), [483] equation (2), [482] equations (4) (5) for examples of implicit Poincaré maps as in (7.42).

\(^{17}\) Apart from the examples presented here, the interested reader may have a look at [392] equation (1.3), [295] equation (1.3), [574] equations (24)-(29), [431] equation (4), [483] equation (2), [482] equations (4) (5) for examples of implicit Poincaré maps as in (7.42).

\(^{18}\) Let $f : x \in \mathbb{R}^n \mapsto f(x) \in \mathbb{R}^m$. Recall that $\frac{\partial f}{\partial x}(x_0) = \nabla_x f(x_0) \in \mathbb{R}^{m \times n}$ denotes the gradient of $f$ at $x_0$, so that the Jacobian is given by $D_x f = \frac{\partial f}{\partial x} \in \mathbb{R}^{m \times n}$. If $g = foh$ with $h : \mathbb{R}^n \mapsto \mathbb{R}^p, f : \mathbb{R}^p \mapsto \mathbb{R}^k$, then $\nabla_x g(x_0) = \nabla_x h(x_0) \nabla_y f(y_0)$ and $D_x g(x_0) = D_y f(y_0) D_x h(x_0)$, where $y_0 = h(x_0)$.
(stability of periodic motions of an ellipsoid of revolution colliding with a fixed smooth plane) \[239\] (studies trajectories which attain the constraint tangentially, i.e. collision free trajectories, and their stability) \[235\] (uses the Zhuravlev-Ivanov nonsmooth coordinate change to study the local stability via linearization, of fixed point and periodic trajectories) \[230\] \[238\] (orbital stability of periodic motions of \(n\)-degree-of-freedom systems with \(T_L(t_k) = 0\) and a codimension one unilateral constraint; use of Lyapunov's holomorphic integral theorem \(^{19}\) to prove via local arguments the existence of periodic trajectories), see also \[292\].

**Remark 7.10** When the free-motion vector field is conservative, motions with collisions can be periodic only if \(T_L(t_k) = 0\). Otherwise periodic motions must be collision-free.

### The bouncing-ball example

The benchmark example of the bouncing ball has been thoroughly studied. It may be seen as a simplified version of the Fermi accelerator model (A ball bounces between a fixed and a moving wall) \[210\]. The model of the bouncing ball when the table is moving is often simplified in order to explicitly get a two-dimensional Poincaré map from (7.42). In fact all the external effects acting on the table are neglected to consider the table's motion, so that it simply appears in the model as a time-dependent unilateral constraint (Such assumption has also been made in some studies on juggling robots that we describe in remark 7.13). Let the table motion be given by \(x_2(t) = -A \sin(\omega t)\), while \(x_1\) denotes the ball's position. Here one does not care about how such motion may be created, by which control. The assumption that the mass of the table is infinite (equivalently that of the ball is close to zero) allows to disregard the effects of the shocks on its velocity. Assume further that \(t_{k+1} - t_k = \frac{2\omega_1(t_k^\star)}{g}\), and that \(\dot{x}_1(t_{k+1}^\star) = -\dot{x}_1(t_k^\star)\). These assumptions are satisfied if it is supposed that the ball strikes the table at the same height at each impact \(^{20}\), and that the displacements of the table are negligible compared to those of the table. In other words although the true dynamics cannot be explicitly solved, such hypotheses allow one to approximate (7.42). Then it is possible to derive the following impact map:

\[
\begin{align*}
\Phi_{k+1} &= \Phi_k + \dot{x}_k \\
\dot{x}_{k+1} &= \epsilon \dot{x}_k - \gamma \cos(\Phi_k + \dot{x}_k)
\end{align*}
\]

where \(\dot{x}_k = \frac{2\omega_1(t_k^\star)}{g}\), \(\Phi_k = \omega t_k\), and \(\gamma = \frac{2\omega^2(1+\epsilon)A}{g}\).

---

\(^{19}\)In case of Hamiltonian (i.e. conservative) systems with an analytic Hamiltonian function, Lyapunov's holomorphic integral theorem states that for every pair of pure imaginary roots \(\pm j\lambda\) of the system's characteristic equation, and when there are no other roots, a family of periodic solutions exists whose period tends to \(\frac{2\pi}{\lambda}\) as their amplitude tends to zero.

\(^{20}\)This, in case of feedback control of a juggling robot, should be guaranteed by the controller, but not a priori supposed.
It is noteworthy that since the system is nonautonomous, the Poincaré impact map must explicitly contain the time: indeed the flight-times \textit{a priori} depend on the exogeneous excitation of the table and are not only a function of the postimpact state values. Such a map can be shown to possess a complex dynamical behaviour, see e.g. [142] [181] [158] [209] [30] [293] [290] [354] [370] [354] [447] [446] [482] [537] [547] [584] [590]. But if \( e < 1 \), the velocity remains bounded and there exists a trapping region in the plane \((\Phi, \dot{x})\) [181], which hampers unboundedness results as in theorem 8.1 in chapter 8. Notice that these studies prove that the time-dependence of the constraints \( f(q, t) < 0 \) has a great influence on the system's dynamics (which is not so apparent by comparing (7.19) to (7.44), but it is a property of certain apparently simple systems to have a complex "hidden" behaviour). It is therefore not gratuitous to assume they are time-independent.

Remark 7.11 One may wonder what is the influence on the predicted dynamical behaviour of the various assumptions done to derive the Poincaré mapping in (7.44), namely that the ball strikes always at the same height, and that the table displacements are negligible with respect to the ball displacements. It has been shown in [30] that for high enough restitution coefficient, the approximation does not result in significant deviance in the behaviour. However the assumption that the table is much heavier than the ball is more stringent. Indeed if both masses are comparable, the "table" will suffer from velocities discontinuities at each impact. How does this influence the overall motion of the system? Moreover from a feedback control perspective (juggling), this will complicate the problem: in particular the table can no longer possess a sinusoidal motion due to the velocity jumps. One has to design a controller for the table that insures that at each impact, the table has the right velocity. Velocity discontinuities have to be incorporated into the control design.

Additional comments and studies

The major difficulty in studying impacting systems is that in general, not only are the impact Poincaré maps difficult to obtain explicitly (but this is not specific to those systems), but they are of dimension \( \geq 2 \). Hence all the tools that apply to one-dimensional applications [208] like for instance the celebrated Sarkovskii's ordering theorem, do not apply. Pioneering works on the dynamics of simple impacting systems can be found in [311] [179] [180] [18] [576] [326] [312] [470] [133] [140] [347]. One of the first paper containing a study on existence of periodic trajectories and their stability was written by Masri and Caughey [347], that we described above. More recently, simple impacting devices (impacting oscillator, damper ...) dynamical study has received attention in numerous works, see e.g. [80] [81] [107] [199] [191] [229] [257] [265] [258] [237] [335] [336] [329] [339] [617] [238] [239] [240] [287] [338] [339] [340] [401] [236] [262] [483] [475] [487] [485] [486] [401] [431] [581] [582] [484] [552] [553] [348] [121] [578] [566]. Experimental results are presented in [370]. Concerning billiards (see definition 7.3) we refer the reader to the mathematical book by Kozlov and Treshchev [292] (see also [585] [47] [493] [494] and references therein), where
many Russian references can be found concerning the dynamical analysis of such systems. The works on dynamics of impacting simple systems were motivated by applications (mechanisms with clearance, see [256] [184] [548] like gearboxes [210] [263] [474] [254] [448] [255] and references therein (let us also mention the work of A.E. Kobrinskii cited in [31] on dynamics of systems with clearances)(21), the use of pin joints in space truss structures and chaotic dynamics of these systems [369] [310], drilling machines [171], motion of fluid tanks [395], motion in print hammers [193] and have been verified experimentally, see e.g. among others [370] [290]. Applications can also be found in practical devices used to generate aerosol streams: the model is a particle bouncing between two charged diverging plates, see [54] §9.1 and [405]. The French company Electricité de France (EDF) also conducts research on the dynamics of assembly devices where impacts play an important role [417] chapter 2. Chaos in impacting systems is apparently not restricted to values of \( e \) close to 1, but can appear for inelastic impacts (\( e = 0.6 \) in [431]). As we already announced, we do not describe in more details these works, but it is worth being aware of the numerous applications of percussive dynamics. The reader is also referred to the survey [481] on chaotic dynamics in mechanical systems, with or without impact phenomena. Let us finally mention the study in [286] about the dynamical behaviour of \( n \)-degree-of-freedom systems with smooth unilateral constraints, \( T_L(t_k) = 0 \), and acted upon by external forces of stochastic nature.

Remark 7.12 It is clear that similarly as for smooth dynamics, the study of impacting systems relies heavily on Poincaré maps. One could therefore think that such systems are just another particular case of dynamical systems, and that all the general tools developed for smooth systems apply directly. This is not the case. Indeed, impacting systems possess particular features. For instance, the impact Poincaré map can have a singular Jacobian at certain points [484] [485] [582]. This is related to so-called \( C \)-bifurcations, which occur when the system evolves from a free regime of motion to an impacting regime, through a grazing trajectory (more precisely, a \( C \)-bifurcation occurs in a configuration such that \( f(q_0) = 0 \), \( f(q_0) = 0 \) and the normal component of the force points inwards the admissible domain \( \Phi \)). In this case the system may evolve from an impact-free periodic motion to an impacting motion, and there corresponds a bifurcation to this particular evolution. The section map \( P \) can possess a discontinuity at such a point. This has received attention in [80] [81][106] [146] [147] [148] [149] [235][301] [401] [402] [237]. When the parameters of the system are varied (for instance the forcing term magnitude or frequency), the trajectories' behaviour can change suddenly at the grazing trajectory. A periodic trajectory can be changed into chaotic motion, followed by a period adding cascade, or the creation of a large number of periodic trajectories. Let us mention that Ivanov [237] studies this problem using an approximating problem \( P_n \) and results on convergence of solutions of \( P_n \) [292] [291], which once again proves the great usefulness of the path consisting of replacing the rigid stops by

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21The study of rattling-noise in gearboxes is fundamental to reduce the noise level and the vibrations in cars.
compliant ones. Also Chin et al [106] observe 3 major types of C-bifurcations in a simple impact oscillator: from stable period-1 orbits to a reversed infinite period adding cascade, or to attracting chaos occupying a full interval of the bifurcation parameter, and collision of an unstable maximal periodic orbit and a period-1 orbit. They all are unconventional, in the sense that they do not occur in smooth systems. These authors conclude that it is expected that those bifurcations are universal for impacting systems.

Related work on the behaviour of a free system consisting of two bodies joined by a spring, and striking a rigid surface, can be found in [392]. The authors study the influence of the mass ratio and of the restitution coefficient on the number of impacts, using a technique developed in [391] on the approximation of the discrete-time system in (7.42).

Remark 7.13 "Time-discretization" of simple systems has also been investigated in the framework of juggling robots [83] [571] [620] [555] [556]. A simple one-degree-of-freedom juggling robot possess the following dynamics

\[
\begin{align*}
    m_1 \ddot{y}_1 &= -m_1 g \\
    m_2 \ddot{y}_2 &= -m_2 g + u
\end{align*}
\] (7.45)

with the constraint \( y_1 - y_2 \geq 0 \). It is apparent from (7.45) that the only link between the two subsystems (robot with position \( y_2 \) and ball or object with position \( y_1 \)) is obtained through the restitution rule and the impact dynamics, i.e.

\[
\begin{align*}
    y_2(t_k^-) &= -e \left( \dot{y}_1(t_k^-) - \dot{y}_2(t_k^-) \right) \\
    m_1 \sigma y_1(t_k) + m_2 \sigma y_2(t_k) &= 0 \\
    y_1(t_k) &= y_2(t_k)
\end{align*}
\] (7.46)

The basic idea is therefore to control the first subsystem (the robot) in order to inject into the second one (the ball) a correct fictitious input. It is apparent from (7.46) that the only variable on which we can play to influence the dynamics of the ball is the robot preimpact velocity \( \dot{y}_2(t_k^-) \). This is depicted in figure 7.3. The position \( y_2(t_k) \) in turn fixes the collisions positions. Hence if one is able to obtain \( \dot{y}_2(t_k^-) \rightarrow \dot{y}_2^* \) and \( y_2(t_k) \rightarrow y_2^* \) for some arbitrary desired values \( \dot{y}_2^* \), \( y_2^* \) (notice that \( \dot{y}_2^* \) will \textit{a priori} be a function of \( \dot{y}_1(t_k^-) \) and system parameters), it is possible to drive \( \dot{y}_1(t_k^+) \) and \( y_1(t_k) \) towards arbitrary desired values. The whole problem is to derive a bounded controller \( u \) (\textit{a priori} a function of positions and velocities of both systems), such that this goal is achieved. This is not an obvious task, especially when both systems' velocities possess discontinuities. One may start from the aim (motion of the ball) and obtain a discrete-time operator \( (\dot{y}_1(t_k^+), \Delta_k) \rightarrow (\dot{y}_1(t_{k+1}^+), \Delta_{k+1}) \) with the desired properties. A simplified point of view is to assume that \( m_2 \) is
Figure 7.3: Block-diagram of a simple juggling robot

much larger than $m_1$, so that $\dot{y}_2$ remains continuous at the impact times\(^{22}\). This
direction is taken in [83] [458]. The obtained discrete system is nonlinear in the
inputs. The design of the actual continuous time input such that convergence of
the fictitious inputs towards their desired values is guaranteed is apparently a hard
task in general and has not been satisfactorily answered to yet. This manner of
solving the juggling control problem is quite similar as treating the impacts as pure
exogeneous signals, and then investigating which kind of controlled mechanism may
be used to guarantee that the sequences $\{t_k\}$ and $\{p_k\}$ are effectively realized, using
physical restitution models. Connected work can be found in [347] [264] [265] who
study the effect of an impacting device on the dynamics of a quite simple system.
Limit cycles and other periodic trajectories are created by suitably choosing the
impact times, see also Russian references in [27]. It is clear that it would also be
quite interesting to connect this with the studies on the bouncing ball dynamics,
see (7.44). A work in this direction can be found in Vincent [555] who basically
uses both the chaotic and the periodic orbits of the mapping in (7.44) to stabilize
the juggler: the robot has a sinusoidal motion, and chaotic dynamics are created
and used to bring the system in the basin of attraction of the desired limit cycle of
the ball. Then the controller is switched to make the robot oscillate sinusoidally at
a different frequency, that corresponds to the limit cycle in question. This way of
proceeding (using a controller to bring the system into the basin of attraction of a
locally stable controller via switching and a hybrid controller) has also been used for
other sorts of systems (see [512]). One drawback of the control strategy proposed
in [555] is that no information is given on the transient: when will the orbit of the
ball attain the basin of attraction of the limit cycle?

The problem of controlling an impacting system as simple as a one-degree-of-
freedom juggler (in particular without assuming $\frac{m_1}{m_2} = 0$) can therefore be considered
to be still open.

Remark 7.14 As we pointed out earlier, the coordinate change proposed in [235]
can be used to transform an impacting system into a variable-structure-like system.
The system in (7.19) becomes trivially (here $f(t, x, \dot{x}) = -g$):

\[
\begin{align*}
\dot{s} &= R\nu \\
\dot{\nu} &= -gR^{-1}\text{sgn}(s)
\end{align*}
\]

\(^{22}\)Note that such an assumption may not at all be verified in certain cases, for instance if the
ball is an iron sphere instead of a ping-pong ball.
and \( \dot{s} = \dot{v} = 0 \) when \((s, v) = (0, 0)\). Then stability of trajectories can be studied \textit{via} smooth or nonsmooth Lyapunov functions [154] [489] that make use of Clarke's generalized gradient (see appendix D), but it remains to be proved that in general such functions exist. It is easily checked from (7.47) that for \( e = 1 \), then \( R = 1 \) and the trajectories are closed and composed of portions of parabolas. The tangent vector to the trajectories switches from \((v, g)\) to \((v, -g)\) when they cross \( s = 0 \). Notice that (5.42) (7.19) (7.47), the variable changes considered in example 1.3 and the formulation in [416] provide us with numerous different ways of representing the same dynamical system as simple as the bouncing ball!

### 7.1.5 Conclusions

In conclusions on this section, the available stability results in the literature do not permit to really analyze the stability of impacting systems. This is mainly due to some basic theoretical reasons for Lyapunov stability applied to systems subject to unilateral constraints (see subsection 7.1.2), or because only the impacting phase is considered and analyzed as a discrete time system (Poincaré impact map): it is necessary to merge both approaches in an appropriate way to define and study for instance stability of manipulators during a complete robotic task. Preliminary work in this direction is proposed in chapter 8. Basically stability can be attacked \textit{via} the separation of smooth and nonsmooth dynamics, and with a unique Lyapunov-like function for both parts. The central fact that the trajectories are \( RCLBV \) implies a countable set of discontinuities (see appendix C), hence the nonsmooth part is well defined as a discrete-time-like system. Note that if the trajectories were \textit{everywhere} discontinuous, stability would require the preliminary redefinition of the notion of a trajectory (functions of \( RCLBV \) would not be sufficient) and the tools we develop would no longer be applicable. A subject of future research work is also to investigate whether all the above cited results on dynamical behaviour of impacting systems may be used to study the stabilization and the closed-loop behaviour of a controlled manipulator striking a possibly dynamic environment (in chapter 8 only the case of a static environment is analyzed). For instance [560] provided a complete analysis for several "classical" force controllers, considering compliant environments. The extension to the case when percussions are considered and when a rigid bodies shock model is chosen is not trivial, since the dynamics are completely modified by the algebraic impact laws. Given an environment model of the same order of complexity than those in [560] or [360], it would be interesting to know whether the variation of control parameters (for instance a PID force or motion control for a 1-degree-of-freedom robot) yields significant modifications of the closed-loop dynamical behaviour or if asymptotic stability is a generic situation.
7.2 Stability: from compliant to rigid models

The goal of the study in this section is to point out some problems related with stabilization of motion-controlled manipulators that come in contact with a compliant environment (in particular the sufficient conditions guaranteeing asymptotic convergence of the solutions towards the steady-state solution), and to propose a particular stability analysis that applies to both the compliant and the rigid cases. We restrict ourselves to a simple continuous PD motion controller (in contrast with the sophisticated switching controllers studied in [342] [360], see also chapter 8), and to the case of a purely elastic environment (that corresponds to the limit case when the impact itself does not dissipate energy). A question a designer may ask himself when facing a real problem is the following: is the environment to be considered as flexible or as rigid? In general one considers that rigid collisions occur when the bodies show "sufficiently small" deformation so that they are geometrically rigid at a global observation scale [385]. The answer is crucial for the choice of the control algorithm (for instance in [602] it is shown that an integral force feedback helps in stabilizing the impact phase when the environment is (sufficiently) rigid, whereas in [560] it is shown that it is not suitable for a (sufficiently) flexible environment) and depending on it, the analysis of the whole robotic task may be quite different. Note that the boundary between "flexible" and "rigid" is quite clear from a mathematical point of view, but not from a practical one: besides clearly rigid environments made of hard materials (stone, iron ...) and clearly flexible ones, some others might be considered to belong to one class or the other one depending on the task (masses of the bodies that collide, accuracy of the measurements, limits of the actuators ...).

Basically one can treat this problem from different point of views, for instance:

i) Study conditions that guarantee that after the first contact has occured, there is no rebound, see e.g. [342],

ii) Relax the bounceless conditions by studying conditions that insure Lyapunov quadratic stability of the system, i.e. find a Lyapunov function $V(x)$ such that along trajectories of the system $\dot{V}(x) = -x^TQx$ with $Q > 0$ (which do not a priori guarantee that the robot's tip will never take off the environment's surface), see e.g. [360]. Since these tools will generally provide sufficient conditions only, it is worth investigating whether the deduced stability conditions are of any practical interest or not. In particular, if they yield lowerbounds on the feedback gains that are proportional to the environment's stiffness, it is clear that as soon as this stiffness becomes too large, the conditions become useless. It is then natural to seek a convergence proof that is independent of the stiffness as well as of the feedback gains values. It is note worthy that this study can also be seen as a study on conditions of stability of a force control scheme, taking into account the fact that the constraints are unilateral, i.e. the robot's tip may take off the surface and possibly start a sequence of rebounds. Still, another motivation is to investigate whether the solutions of a given sequence of smooth "problems" converges towards a nonsmooth one. Here we simply investigate whether a certain stability property
7.2. STABILITY: FROM COMPLIANT TO RIGID MODELS

for smooth dynamics still holds in the rigid bodies limit. For the sake of briefness, we shall study only approach ii). In fact it can easily (and logically) concluded that bounceless conditions are impossible to obtain with finite force control, for non-zero contact velocity, when the stiffness grows unbounded.

7.2.1 System’s dynamics

The system depicted in figure 7.4 consists of a simple mass moving horizontally without friction whose position is given by $x(t)$ and a compliant environment at $x = 0$ whose model is a spring with stiffness $k > 0$.

The control law is given by $u = -\lambda_2 \dot{x} - \lambda_1 (x - x_d)$, $x_d \geq 0$, $\lambda_1 > 0$, $\lambda_2 > 0$. We assume that contact is established at $t = 0$, with $x = 0$. Then the equations that govern our system are:

$$
\begin{align*}
  m\ddot{x} + \lambda_2 \dot{x} + \lambda_1 x &= \lambda_1 x_d & \text{if } x < 0 \\
  m\ddot{x} + \lambda_2 \dot{x} + (\lambda_1 + k)x &= \lambda_1 x_d & \text{if } x \geq 0
\end{align*}
$$

(7.48)

Although this system consists of two switching vector-fields, the transition between both is continuous: this is called a continuous switching system in [65] (23). If damping is added in the environment model, then the system is no longer a continuous switching system. Notice that convergence of the state $(x, \dot{x})$ towards the fixed point of the second equation in (7.48) may perhaps be investigated by considering the associated equivalent total energy of the closed-loop system in (7.48). However in the sequel we focus on a particular stability property of this equilibrium point. The motivation for studying this type of stability is evident if one thinks of more complicated tasks as considered for instance in [360]. Also the equivalence with a mechanical system may no longer be possible in certain cases, e.g. when the feedback loop contains time-delays. From the mathematical results presented in chapter 2, see theorem 2.1, it follows that as $k \to +\infty$, the solutions of the dynamical system

\[23\] It is also straightforward to apply the extension of the Poincaré-Bendixson’s theorem to continuous switching systems presented in [65] to deduce that the system in (7.48) has no closed trajectories as long as $\lambda_2 > 0$. 

Figure 7.4: Controlled mass colliding an elastic wall.
in (7.48) converge towards the solution of the system:
\[
\begin{cases}
  m\ddot{x} + \lambda_2 \dot{x} + \lambda_1 x = \lambda_1 x_d, & \ddot{x} = \min(0, -\lambda_2 \dot{x} - \lambda_1 x + \lambda_1 x_d) \text{ if } \dot{x}(t_k^+) = 0 \\
  x \leq 0 \\
  \dot{x}(t_k^+) = -\dot{x}(t_k^-) & x(t_k) = 0
\end{cases}
\]
(7.49)

Note that this convergence remains true even if damping is added, introducing a proper restitution coefficient. Our main goal is therefore to look for a stability analysis that is able to encompass both systems in (7.48) and (7.49). Since the solutions $x_n(t)$, $\dot{x}_n(t)$ of (7.48) (with $k = k_n$, \{k_n\} a strictly increasing sequence) tend as $n \to +\infty$ towards a solution of (7.49), there should logically exist a manner to analyze their stability in a common framework.

**Remark 7.15** The switching between two vectors fields in (7.48) is an autonomous switching [63]. Let us take $x_d = 0$ and let us define the matrix
\[
A = \begin{pmatrix} 0 & 1 \\ -\alpha k - \lambda_1 & -\lambda_2 \end{pmatrix}
\]
Then the system can be rewritten as follows
\[
\begin{pmatrix} \dot{x} \\ \dot{\dot{x}} \end{pmatrix} = A \begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \quad \dot{x} = -\text{sgn}(x(t_k^-)) \delta_{t_k}
\]
(7.50)
and with $x(0) > 0$, $\alpha(0) = 1$, $x(t_k) = 0$. Hence we have transformed an autonomous switching into an autonomous jump, by augmenting the state space of the system. The usefulness of this operation remains however unclear for stability analysis and control purposes.

### 7.2.2 Lyapunov stability analysis

In the following we shall analyze the stability of system (7.48) using a single Lyapunov function. To begin with, we show how the stability analysis of the closed-loop system in (7.48) can be led with a particular Lyapunov function candidate: let us consider
\[
V = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} \lambda \ddot{x}^2 + c\ddot{x}
\]
(7.51)
with $\lambda = \lambda_1 + k + \frac{\lambda_2 c}{m}$, $c > 0$ is such that $c^2 - \lambda_2 c - m(\lambda_1 + k) < 0$ (since $\Delta = \lambda_2^2 + 4m(\lambda_1 + k) > 0$, and $\sqrt{\Delta} - \lambda_2 > 0$, such a $c$ can always be chosen arbitrarily small), and $\ddot{x} = x - \frac{\lambda_2 x_d}{\lambda_1 + k}$. $\lambda$ and $c$ guarantee that $V$ is positive definite. Now we get:

- $x < 0$ (non-contact)
\[
\dot{V} \leq (\lambda_2 + c + \frac{1}{2} k_2 + \frac{1}{2} k) \dot{x}^2 + \left(\frac{\lambda_1 k}{m} \left(\frac{\lambda_1 k}{m} x_d \ddot{x} + \frac{1}{2} \left(\frac{\lambda_1 k}{m} x_d \ddot{x} \right)^2 + \frac{1}{2} \left(\frac{c \lambda_1 k}{m (\lambda_1 + k)} \right)^2 \right)
\]
(7.52)
or in compact form

\[ V = -a_{nc}x^2 - b_{nc}\ddot{x}^2 + R \]  

with \( R > 0 \).

- \( x > 0 \) (contact)

\[ \dot{V} = (-\lambda_2 + c)\dot{x}^2 - \frac{\lambda_1 + k}{m}c\dot{x}^2 = -a_c\dot{x}^2 - b_c\dot{x}^2 \]  

(7.54)

**Claim 7.2** For any stiffness \( 0 < k < +\infty \) there exist \( P = P^T > 0, Q = Q^T > 0, \lambda_1 > 0, \lambda_2 > 0 \) such that \( \dot{V} < 0 \) for all \( t \geq 0 \), \( V = z^TPz \leq -z^TQz \). Thus the equilibrium point \( z = 0 \) is globally asymptotically stable (GAS) in the sense of Lyapunov and the system in (7.48) is quadratically stable.

**Proof of claim 7.2**

\( \lambda_1 \) and \( \lambda_2 \) can be chosen such that \( a_{nc} > 0 \) and \( b_{nc} > 0 \). Thus we conclude that for all \( x \): \( \dot{V} \leq -\alpha\dot{x}^2 - \beta\ddot{x}^2 + R \), with \( \alpha = \min(a_{nc}, a_c) \), \( \beta = \min(b_{nc}, b_c) \). Following the arguments in [114], we deduce that the state \((\ddot{x}, \dot{x})\) converges in finite time in a ball with radius \( r \), with \( r \to 0 \) as \( \lambda_1 \) and \( \lambda_2 \) tend to \( +\infty \). Therefore for all \( t \geq \bar{t} \), \( \bar{t} < +\infty \), we get \( |\ddot{x}| < r \). Now notice (see (7.52)) that \( R \to \frac{1}{2}k^2x_0^2(1 + \frac{x_d^2}{x_0^2}) \) as \( \lambda_1 \to +\infty \). Since by taking \( \lambda_1 \) and \( \lambda_2 \) large enough \( r \) can be made arbitrarily small and since \( \frac{\dot{x}^2}{\lambda_1 + k} \to x_d \) when \( \lambda_1 \to +\infty \), it follows that for \( \lambda_1 \) large enough, \( |\ddot{x}(t)| < r \) for \( t \geq \bar{t} \) implies \( x(t) > 0 \) for \( t \geq \bar{t} \). Then (7.54) implies that both \( \dot{x} \) and \( \ddot{x} \) converge asymptotically to zero. Notice that outside some ball \( B_R \) we have for some \( t \) : \( V \leq -z^TQz \), with \( z^T = (x, \dot{x}) \). This can be easily deduced by splitting \( \alpha \) and \( \beta \) into \( \alpha_1 \) and \( \alpha_2 \), \( \beta_1 \) and \( \beta_2 \): then \( \dot{V} \leq -\alpha_1\dot{x}^2 - \beta_1\ddot{x}^2 - \alpha_2\dot{x}^2 - \beta_2\ddot{x}^2 + R \), so that outside the ball \( B_R \) with \( R = \frac{R}{\min(\alpha_1, \beta_2)} \), we get \( \dot{V} \leq -\alpha_1\dot{x}^2 - \beta_1\ddot{x}^2 \). Still \( \lambda_1 \) and \( \lambda_2 \) can be chosen large enough so that \( R \) is as small as desired. Thus we deduce that for \( 0 \leq t \leq \bar{t} \), \( \| z(t) \| \leq \sqrt{\frac{V(0)}{\lambda_{\min}}} \exp(-\frac{\lambda_{\min}Q}{\lambda_{\max}}t) \), i.e. the ball \( B_R \) is reached exponentially fast.

### 7.2.3 Analysis of quadratic stability conditions for large stiffness values

We shall be content with these existence results on the feedback gains in the above analysis. However, let us note that if one takes the sufficient conditions for stability deduced from the above analysis, then the feedback gains \( \lambda_1 \) and \( \lambda_2 \to +\infty \) as \( k \to +\infty \). In other words, the sufficient conditions found imply feedback gains growing without bound as \( n \to +\infty \). This suggests that in order to obtain quadratic Lyapunov stability of (7.48) one has to choose feedback gains proportional to the stiffness \( k \) as \( k \) becomes large. The aim of this section is to prove that this is true. Notice that such a result is not satisfying, because the solutions of (7.49) are bounded and in a sense are Lyapunov stable: indeed as we shall see in chapter 8, the
Poincaré impact map $P_\Sigma$ associated to the system (7.49) and the Poincaré section $\Sigma = \{(x, \dot{x}) : x = 0\}$ is Lyapunov stable. This can be shown by taking $k = +\infty$ for $V$ in (7.51), so that $\ddot{x} = \dot{x}$: then one computes that $V(t_{k+1}^-) - V(t_k^+) \leq 0$ and that $\sigma V(t_k) = 0$, using the fact that $x(t_k) = 0$, $x < 0$ on $(t_k, t_{k+1})$ and that the sequence $\{t_k\}$ exists. Then one notes that the restriction of $V$ to $\Sigma$, i.e. $V_\Sigma$, satisfies these inequalities also since $V_\Sigma(t_k) = V(t_k)$, hence Lyapunov stability of the fixed point $\dot{x}(t_k) = 0$ of $P_\Sigma$ (Although $P_\Sigma$ is not calculable explicitely). Since this stability is obtained for bounded feedback gains, one logically expects to be able to find out a stability criterion that works "uniformly" with respect to $k$.

Let us rewrite (7.48) in state space form as

$$z \in (NC) \triangleq \{x : x < 0\} \quad \dot{z} = A_c z + \left( \begin{array}{c} 0 \\ \frac{k}{m} \end{array} \right)$$

(7.55)

where $z^T = (x - \frac{\lambda_1 x_2}{\lambda_1 + k}, \dot{x})$, $A_c = \left( \begin{array}{cc} -\frac{1}{m} (\lambda_1 + k) & 1 \\ \frac{\lambda_2}{m} & -\frac{1}{m} \end{array} \right)$. Clearly the choice of the first component of $z$ stems from the fact that we want to stabilize the robot in contact with the environment. Moreover from (7.48) one sees that the equilibrium point of the first equation belongs to $(C)$, which means that the system in (7.55) possesses in fact only one equilibrium point, i.e. $z^T = (0, 0)$ (Note that the uniqueness holds for any value of $x_d$; when $x_d = 0$ both equations in (7.48) have the same equilibrium point $(x, \dot{x}) = (0, 0)$). Stability of $A_c$ is independent of $k$ since its eigenvalues are either real strictly negative or with real part equal to $\frac{-\lambda_2}{2m}$. Thus for any $Q_c = Q_c^T > 0$ there always exists $P = P^T > 0$ such that $A_c^T P + P A_c = -Q_c$. Since we want to stabilize the equilibrium point $z = 0$, we choose a Lyapunov function candidate as $V = z^T P z$.

Along trajectories in $(NC)$ we get $\dot{V} = -z^T Q_c z + z^T P \left( \begin{array}{c} 0 \\ \frac{2k}{m} \end{array} \right)$. For simplicity of the analysis, let us choose $x_d = 0$. Then we can write $\dot{V} = -z^T Q_c z + z^T P K z \triangleq -z^T Q_c z$, with $K \triangleq \left( \begin{array}{cc} 0 & 0 \\ \frac{2k}{m} & 0 \end{array} \right)$.

Simple calculations yield:

$$Q_c = \begin{bmatrix} 2 \frac{\lambda_1 + k}{m} p_{12} & \frac{\lambda_2}{m} p_{12} + \frac{\lambda_1 + k}{m} p_{22} - p_{11} \\ \frac{\lambda_2}{m} p_{12} + \frac{\lambda_1 + k}{m} p_{22} - p_{11} & 2 \left( \frac{\lambda_2}{m} p_{22} - p_{12} \right) \end{bmatrix}$$

(7.56)

$$Q_{nc} = \begin{bmatrix} 2 \frac{\lambda_1}{m} p_{12} & \frac{\lambda_2}{m} p_{12} + \frac{\lambda_1}{m} p_{22} - p_{11} \\ \frac{\lambda_2}{m} p_{12} + \frac{\lambda_1}{m} p_{22} - p_{11} & 2 \left( \frac{\lambda_2}{m} p_{22} - p_{12} \right) \end{bmatrix}$$

(7.57)

where $Q_{nc}$ is the symmetric part of the matrix $Q_c$, that is independent of $k$. It is worth noting that only the skew-symmetric part of $Q_c$ depends on $k$.

---

24 This last point will be important to assure via a suitable controller when one wants to stabilize a system on a surface.
Thus a necessary and sufficient condition for $Q_c$ to be positive definite is that:

- $\frac{\lambda_1 + k}{m} p_{12} > 0$
- $\det(Q_c) = 4 \frac{\lambda_1 + k}{m} p_{12} \left( \frac{\lambda_2}{m} p_{22} - p_{12} \right) - \left( \frac{\lambda_2}{m} p_{12} + \frac{\lambda_1 + k}{m} p_{22} - p_{11} \right)^2 > 0$

For $Q_{nc}$ the conditions are the following:

- $\frac{2\lambda_1}{m} p_{12} > 0$
- $\det(Q_{nc}) = 4 \frac{\lambda_1}{m} p_{12} \left( \frac{\lambda_2}{m} p_{22} - p_{12} \right) - \left( \frac{\lambda_2}{m} p_{12} + \frac{\lambda_1}{m} p_{22} - p_{11} \right)^2 > 0$

Our aim in this section is to examine the conditions such that the simple system (7.48) is Lyapunov quadratically stable, and in particular to find out which kind of conditions this implies on the feedback gains. As shown below, the following result is true:

**Claim 7.3** Consider the 1-degree-of-freedom closed-loop equations in (7.48) with $x_d = 0$. Then quadratic stability of the system implies conditions such that when the environment's stiffness $k$ grows unbounded, then the feedback gains $\lambda_1$ and/or $\lambda_2$ have to be chosen of order $\geq k^\beta$, $\beta \geq \frac{1}{2}$ to guarantee that the solution $P$ of the Lyapunov equation remains bounded away from singularities (i.e. $\lambda_{\min}(P) \geq \delta > 0$ for some $\delta$) and that the matrices $Q_c$ and $Q_{nc}$ remain positive definite.

$\lambda_{\min}(P)$ denotes the minimum eigenvalue of the matrix $P$.

**Remark 7.16** As long as $x_d = 0$, the term quadratic stability as defined in [35] is justified since both vector fields in (NC) and (C) have a unique fixed point. We may see the analysis as a robustness analysis of a system where the stiffness parameter $K \in \{0, k\}$. This is no longer the case if $x_d \neq 0$.

**Remark 7.17** Clearly when $x_d = 0$ the system in (7.48) can be analyzed considering the first equation only provided the initial condition $x(0)$ is negative (which is the only possible choice when $k = +\infty$). Then a simple choice of feedback gains implies that the mass never collides with the environment. The above conclusions come from the fact that the analysis is led considering both equations in (7.48), i.e. we require the derivative of a Lyapunov function to be negative definite along two vector fields at the same time.

**Remark 7.18** We have led the calculations assuming that $x_d = 0$ to simplify the analysis. We conjecture that the conclusions hold a fortiori for $x_d > 0$, since stabilization of the equilibrium point $z = 0$ cannot rely on weaker assumptions in this case, as one can easily check in the preceding example (The case $x_d = 0$ is the simplest one when one considers the problem of making $\dot{V}$ above negative definite since in this case both equations have the same equilibrium point).
Remark 7.19 The result of claim 7.3 may seem at first sight surprising: firstly the considered type of stability does not imply that there will be no rebounds, thus one may wonder why such conclusions arise; Secondly as we have already noticed, the stability of the matrix $A_c$ is completely independent of $k$, and one could expect from the Lyapunov theorem that the solutions to the Lyapunov equation should not depend on $k$ either. Note that if we consider the contact task alone, then from the example of a particular Lyapunov function function given in the foregoing section, we have $P = \begin{pmatrix} \frac{\lambda_1 + k}{2} & \frac{c}{2} \\ \frac{c}{2} & \frac{\lambda_2}{2} \end{pmatrix}$ with $c > 0$ bounded such that $P > 0$ for any $k$, that results in $Q_c = \begin{pmatrix} \frac{\lambda_1 + k}{m} & 0 \\ 0 & \lambda_2 - c \end{pmatrix} > 0$ for any $k$. Such conclusions cannot be drawn when both vector fields in (7.48) are considered in the stability analysis, since $\lambda_{\min}P$ approaches zero for bounded gains as $k$ grows unbounded. Note however that from the estimation ($E$):

$$||z||^2 \leq \frac{\lambda_{\max}P}{\lambda_{\min}P} ||z(0)||^2 e^{-\frac{\lambda_{\min}Q_c}{\lambda_{\max}P} t}$$  \hspace{1cm} (7.58)

one would tend to conclude also that $\lambda_2$ has to be proportional to $k^2$ to guarantee a bounded state, one possibility to avoid this being that the ratio $\frac{\lambda_{\max}Q_c}{\lambda_{\min}P}$ increases at least as $\frac{\lambda_{\max}Q_c}{\lambda_{\min}P}$ so that the right-hand side of ($E$) remains bounded. We do not investigate this path and we reiterate that the only thing we do is to study conditions on the feedback gains such that the Lyapunov equation $A_c^T P + PA_c = -Q_c$ possesses a solution $P$ that is bounded-away from singularities and guarantees $Q_c > 0$, $Q_{nc} > 0$.

• Proof of claim 7.3

Starting from the Lyapunov equation, one may first fix $Q_c$ as a positive definite matrix and then try to calculate the unique corresponding positive definite $P$ (see e.g. [554] lemma 42, chapter 5). A second way to attack the problem is to pick a $P > 0$ and study the properties of the resulting $Q$ ([554] p.198). In fact, instead of choosing a $Q_c > 0$ and solving the Lyapunov equation for $P$, we rather consider a matrix $P$ and find conditions such that the corresponding $Q_c$ is positive definite, together with $Q_{nc}$. Thus we prove that the only way for $P$ not to tend towards a singular matrix while keeping $Q_c > 0$ and $Q_{nc} > 0$ when $k$ increases is to take the gain $\lambda_1$ of the same order as $k^2$.

The above determinants can be written in the following way :

$$det(Q_{nc}) = \frac{4\lambda_1}{m} (p_{11}p_{22} - p_{12}^2) - (\frac{\lambda_1}{m} p_{22} + p_{11} - \frac{\lambda_2}{m} p_{12})^2$$

$$det(Q_c) = \frac{4\lambda_1 + k}{m} (p_{11}p_{22} - p_{12}^2) - (\frac{\lambda_1 + k}{m} p_{22} + p_{11} - \frac{\lambda_2}{m} p_{12})^2$$

$$= -\frac{1}{m^2} (\lambda_1 p_{22} + mp_{11} - \lambda_2 p_{12})^2 - 2 \frac{k^{p_{22}}}{m^2} (\lambda_1 p_{22} + mp_{11} - \lambda_2 p_{12})$$

$$+4 \frac{\lambda_1 + k}{m} (p_{11}p_{22} - p_{12}^2) - \left( \frac{k^{p_{22}}}{m} \right)^2$$

The above determinants can be written in the following way :
Let us denote \( y = \frac{A}{P_{22}} + m_{n} \) and \( [P] = P_{11}P_{22} - P_{21}P_{12} \) then:

- \( \det(Q_{nc}) > 0 \Leftrightarrow 4\lambda_{1}m_{1}|P| < Y^{2} > 0 \)
- \( \det(Q_{c}) > 0 \Leftrightarrow Y^{2} + 2kp_{22}Y - 4(\lambda_{1} + k)m_{1}|P| + (kp_{22})^{2} < 0 \)

We deduce that \( Y \) has to satisfy the following inequalities:

\[
\begin{align*}
-2\sqrt{m_{1}|P|} < Y < 2\sqrt{m_{1}|P|} \\
-kP_{22} - 2\sqrt{m_{1}(\lambda_{1} + k)|P|} < Y < -kP_{22} + 2\sqrt{m_{1}(\lambda_{1} + k)|P|}
\end{align*}
\] (7.59)

Since \(-kP_{22} - 2\sqrt{m_{1}(\lambda_{1} + k)|P|} < -2\sqrt{m_{1}|P|}\) there exists a solution for \( Y \) if and only if \(-2\sqrt{m_{1}|P|} < -kP_{22} + 2\sqrt{m_{1}(\lambda_{1} + k)|P|}\), which is found after some manipulations to be equivalent to the following conditions:

\[
\begin{align*}
2\Lambda_{k}^{2}p_{11} - 2\sqrt{\Lambda_{k}^{2}(\Lambda_{k}^{2}p_{11}^{2} - p_{12}^{2})} < p_{22} < 2\Lambda_{k}^{2}p_{11} + 2\sqrt{\Lambda_{k}^{2}(\Lambda_{k}^{2}p_{11}^{2} - p_{12}^{2})} \\
p_{12} < \Lambda_{k}p_{11}
\end{align*}
\] (7.60)

with \( \Lambda_{k} = \sqrt{\frac{m_{1} + k}{k}} \)

Notice that by choosing \( p_{22} = 2\Lambda_{k}^{2}p_{11} \) we can find \( P \) that satisfies (7.60) and that is positive-definite.

From (7.59), \( Y \) satisfies the following inequalities:

\[-2\sqrt{m_{1}|P|} < Y < \min(-kP_{22} + 2\sqrt{m_{1}(\lambda_{1} + k)|P|}, 2\sqrt{m_{1}|P|})\] (7.61)

It is easy to prove that \( \lambda_{\min}(P) \leq p_{22} \) and \( \lambda_{\max}(P) \geq p_{11} \) from which we deduce that

\[
\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \geq \frac{p_{11}}{p_{22}}
\] (7.62)

From (7.60), we can write \( p_{22} < 4\Lambda_{k}^{2}p_{11} \), thus \( P \) has bounded entries when \( p_{11} \) is bounded and the above conditions fulfilled. Then if \( p_{11} \) is a finite real number the conditions of existence of \( Y \) imply that the coefficients \( p_{12} \) and \( p_{22} \) tend to zero when the stiffness of the environment becomes infinite, rendering the matrix \( P \) singular. Let us note that the stability analysis then becomes asymptotically (i.e. when \( k \to +\infty \)) meaningless since \( Q_{nc} \) in (7.57) has bounded entries. The only way to avoid this problem is to increase the gain \( \lambda_{1} \) such that the coefficient \( \Lambda_{k} \) does not tend towards zero when the stiffness increases, i.e. \( \lambda_{1} \) has to be chosen of order \( \geq k^{2} \). Assume that this is done so that \( P \) is well conditioned, and let us examine how \( \lambda_{2} \) has to be chosen. \( \lambda_{2} \) may be found by using (7.61):

\[
\frac{mp_{11} + \lambda_{1}p_{22} - Y_{\max}}{p_{12}} < \lambda_{2} < \frac{mp_{11} + \lambda_{1}p_{22} + 2\sqrt{m_{1}|P|}}{p_{12}}
\] (7.63)
where \( Y_{\text{max}} = \min(-kp_{22} + 2\sqrt{m(\lambda_1 + k)}|P|, 2\sqrt{m\lambda_1}|P|) \). This implies that when \( \lambda_1 \) is of order \( k^2 \) and \( k \) grows unbounded, the gain \( \lambda_2 \) becomes infinite too.

Let us examine what happens if we allow \( p_{11} \) to be proportional to \( k^\alpha, \alpha > 1 \). Then \( p_{22} \) may be chosen of order \( k^{\alpha-1} \) from (7.60). Also \( p_{12} \) will be of order \( \geq k^{\alpha-\frac{1}{2}} \) from the second condition in (7.60). Now from (7.63) we have the following:

1. If \( Y_{\text{max}} \leq 0 \) then obviously \( \lambda_2 \) is of order \( k^{\frac{1}{2}} \) as \( k \to +\infty \). If \( Y_{\text{max}} > 0 \), let us examine the case when \( Y_{\text{max}} = 2\sqrt{m\lambda_1}|P| \); this value is maximum when \( p_{12} \) is minimum, hence bounded, and when both \( p_{11} \) and \( p_{22} \) are maximum, i.e. respectively of orders \( k^\alpha \) and \( k^{\alpha-1} \); then \( Y_{\text{max}} \) is of order \( k^{\alpha-\frac{1}{2}} \) so that \( \lambda_2 \) grows as \( k^{\frac{3}{2}} \). Now if \( Y_{\text{max}} \Delta A = -kp_{22} + 2\sqrt{m(\lambda_1 + k)}|P| \) that we assume \( > 0 \); then necessarily since \( p_{22} \geq 0 \), the second term in \( A \) is at least of the same order as \( kp_{22} \) in \( k \) as \( k \) grows unbounded. Thus at most the order of \( Y_{\text{max}} \) will be that of the second term \( 2\sqrt{m(\lambda_1 + k)}|P| \), which is found to be at most \( k^\alpha \). But if this is the case then this term dominates \( 2\sqrt{m\lambda_1}|P| \) and asymptotically (in \( k \)) \( Y_{\text{max}} \) will necessarily be equal to this last term, hence we are back to the previous case. Now if the order of \( A \) is \( k^\gamma \) with \( \gamma < \alpha \) then \( p_{11} \) will asymptotically dominate \( Y_{\text{max}} \) and the left-hand-side of (7.63) is asymptotically of order \( k^{\frac{3}{2}} \). Thus \( \lambda_1 \) may be chosen bounded but \( \lambda_2 \) will grow unbounded to guarantee \( \lambda_{\text{min}}P \geq \delta > 0 \) for any arbitrarily small but fixed \( \delta \) and \( Q_c > 0, Q_{nc} > 0 \).

### 7.2.4 A stiffness independent convergence analysis

In the following, we propose a convergence analysis different from the one in subsection 7.2.3 to prove that the equilibrium point of (7.48) is asymptotically reached for any initial condition and any value of the feedback gains, independently of the value of \( k \); the particular feature of the analysis is that it extends naturally to the rigid environment case (i.e. \( k = +\infty \)), contrarily to the foregoing one. Roughly speaking, we consider a particular section of the phase-plane, \( x = 0 \). Then we analyze the mass velocity at the instants \( t_i \) when the trajectories cross this section\(^{25}\).

We use the fact that these times define a sequence along which the kinetic energy is non-increasing. It follows that if \( \{t_i\} \) is an infinite sequence, the velocity must converge to zero when \( i \to +\infty \). This leads to a contradiction and there is a finite number of bounces, so that both \( \dot{x} \) and \( x \) converge to zero.

We assume that the mass makes contact with the environment at \( t = t_i \), loses contact at \( t = t_{i+1}, i \in \mathbb{N} \), and that contact occurs at \( x = 0 \). Thus contact occurs on intervals \([t_{2i}, t_{2i+1}]\), and free motion on intervals \([t_{2i+1}, t_{2i+2}]\). Let us consider the positive definite functions

\[
V_c = \frac{1}{2}mx^2 + \frac{1}{2}(\lambda_1 + k) \left( x - \frac{\lambda_1x_d}{\lambda_1 + k} \right)^2 \tag{7.64}
\]

\(^{25}\)We do not use the notation \( t_k \) because the \( t_i \)'s may correspond to detachment. In fact if contact is made at \( t_{2i} \) and lost at \( t_{2i+1} \), then \( t_k = t_{2i} \).
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and

\[ V_{nc} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \lambda_1 (x - x_d)^2 \]  

(7.65)

On intervals \([t_{2i}, t_{2i+1}]\), \(V_c = -\lambda_2 \dot{x}^2\). On intervals \([t_{2i+1}, t_{2i+2}]\), \(V_c = -\lambda_2 \dot{x}^2\). Let \(T(t)\) denote the system's kinetic energy. From the fact that \(V_c(t_{2i+1}) - V_c(t_{2i}) = T(t_{2i+1}) - T(t_{2i})\) and \(V_{nc}(t_{2i+2}) - V_{nc}(t_{2i+1}) = T(t_{2i+2}) - T(t_{2i+1})\), we deduce that for all \(i\), \(T(t_{i+1}) - T(t_i) < 0\), hence \(\lambda |\dot{x}(t_{i+1})| < |\dot{x}(t_i)|\). The same inequalities hold for \(V_c\) and \(V_{nc}\). Now notice that there are two situations: either the sequence of instants \(t_i\) is finite (the bounces stop after a finite time \(t_{2N}\), \(N < +\infty\), and since \(x_d > 0\), \(x(t) > 0\) for all \(t > t_{2N}\)), or this sequence is infinite \(i.e. N = +\infty\).

- If \(N < +\infty\), then for \(t > t_{2N}\) the system is governed by the second equation in (7.48) (indeed each time the mass is "outside" the environment it necessarily collides again after a finite time) and we conclude that \(x \to \frac{\lambda |\dot{x}|}{\lambda_1 + \epsilon} \dot{x} \to 0\) asymptotically, globally and uniformly.

- Assume that \(N = +\infty\). Since the kinetic energy is a positive definite function of the velocity that is non-increasing at times \(t_i\), \(T(t_i)\) converges as \(i \to +\infty\), and so does \(\dot{x}(t_i)\). Suppose that \(|\dot{x}(t_i)| \to |\dot{x}_{ss}|\) with \(|\dot{x}_{ss}| \geq \delta > 0\) (the \(ss\) subscript is for steady state value). Now \(\delta > 0\) and since \(\text{sgn}(\dot{x}(t_i)) = -\text{sgn}(\dot{x}(t_{i+1}))\), \(x(t_i) = x(t_{i+1}) = 0\), the length of the orbit between \(t_i\) and \(t_{i+1}\) is strictly positive. Since the flow of both equations in (7.48) is exponential and bounded for bounded feedback gains and stiffness \(k\), clearly \(\lambda |t_{i+1} - t_i| \Delta \mu_{i+1} > 0\) and \(T(t_i) - T(t_{i+1}) \Delta \beta_{i+1} = \lambda_2 \int_{t_i}^{t_{i+1}} \dot{x}^2 dt > 0\). Note that for fixed and bounded coefficients in (7.48) \(\mu_i\) and \(\beta_i\) depend only on \(\delta\) (the other "initial" condition on the position remaining fixed at the times \(t_i\)) so that in particular \(\beta_i \geq \beta(\delta) > 0\) for all \(i \geq 0\) and \(\delta > 0\). Since \(T(t_i)\) is non-increasing, its limit value is its minimum value and for all \(i \geq 0\), \(|\dot{x}(t_i)| \geq |\dot{x}_{ss}| > \delta\). From the strictly positive variation of the kinetic energy we deduce that \(\dot{x}^2(t_{i+1}) = \dot{x}^2(t_i) - \frac{2\lambda}{m} \mu_i\), so that \(\dot{x}^2 = \dot{x}_0^2 - \sum_{j=0}^{i-1} \beta_j\). Therefore from the fact that the \(\beta_i\)’s are strictly positive, we deduce that \(\dot{x}(t_i)\) cannot converge to a strictly positive \(|\dot{x}_{ss}|\). Since however \(T(t_i)\) and thus \(\dot{x}(t_i)\) converge, we deduce that the only possible limit value for the velocity is \(\dot{x}_{ss} = 0\). (Notice that if \(\delta = 0\), then both \(\mu_i\) and \(\beta_i\) may asymptotically take arbitrarily small values and \(\dot{x}_{ss}^2 = \dot{x}_0^2 - \sum_{j=0}^{i-1} \beta_j\) no longer leads to a contradiction.) Thus we have shown that if there is an infinite number of bounces, then the value of the velocity when contact is established or lost \((x(t_i) = 0)\) is bounded and tends to zero.

Let us now consider an arbitrarily large integer \(i\) such that \(|\dot{x}(t_i)| \) is arbitrarily small, or in other words, for any \(\varepsilon > 0\), there exists \(N(\varepsilon) > 0\) such that \(i > N\) implies \(|\dot{x}(t_i)| < \varepsilon\). We shall denote \(\Delta_i \Delta t_{i+1} - t_i\). First note that from any of the two dynamic equations in (7.48) we get \(\Delta_i \leq \Delta_{\max} < +\infty\) for some \(\Delta_{\max}\) since the "initial" velocities at times \(t_i\) are bounded and tend towards zero. Now we use the fact that both vector fields in (7.48) are explicitly integrable; assume that we place ourselves at \(t_2\) such that \(\dot{x}(t_2) = \varepsilon > 0\), hence the system is in a contact phase for some time since \(\dot{x}(t_i) = \lambda_1 x_d - \lambda_2 \varepsilon > 0\) for some \(\varepsilon > 0\). We thus consider the second equation in (7.48); If the negative roots \(r_1\) and \(r_2\) of the characteristic equation are real and separated, \(r_1 < r_2\), then the solution can be expressed as (recall that
\[ x(t_i) = 0 \text{ for all } i: \]
\[ x(t) = \gamma_1 e^{r_1(t-t_{2i})} + \gamma_2 e^{r_2(t-t_{2i})} + \bar{x}_d \]
(7.66)

with \( \bar{x}_d = \frac{\Delta x_{2i}}{\lambda_1 + \Phi} \), and \( \gamma_1 = -\gamma_2 - \bar{x}_d, \gamma_1 r_1 = -\gamma_2 r_2 + \varepsilon \). Since we assume \textit{a priori} that the sequence \{\( t_i \)\} is infinite, \( t_{2i+1} \) exists and from (7.66) we get
\[ \dot{x}(t_{2i+1}) = \gamma_1 r_1 (e^{r_1 \Delta_{2i+1}} - (1 - \varepsilon)e^{r_2 \Delta_{2i+1}}) \]
(7.67)

From the monotonicity of \{\(|\dot{x}(t_i)|\)\} and its convergence, we deduce that \(|\dot{x}(t_{2i+1})| \leq \varepsilon\). Assume now that the sequence \{\( \Delta_{2i+1} \)\} does not converge towards zero, i.e. there exists \( \Delta > 0 \) such that \( \Delta_{2i+1} \geq \Delta \) for all \( i \). Then we get for any \( \varepsilon > 0 \):
\[ |(1 - \frac{\varepsilon}{\gamma_1 r_1})e^{(r_2 - r_1)\Delta_{2i+1}} - 1| \geq \eta(r_1, r_2, \Delta) > 0 \]
(7.68)

and \( e^{\gamma_1 \Delta_{2i+1}} \geq \kappa(\Delta_{\text{max}}, r_1) > 0 \). From (7.67) we get:
\[ |\gamma_1 r_1 \kappa(\Delta_{\text{max}}, r_1)\eta(r_1, r_2, \Delta)| < \varepsilon \]
(7.69)

which cannot be true for \( \varepsilon \) small enough (Note that the roots as well as \( \Delta \) and \( \Delta_{\text{max}} \) do not depend on \( \varepsilon \)). Since \( \varepsilon \) is arbitrarily small, we deduce that \( \Delta_{2i+1} \to 0 \) as \( i \to +\infty \). A quite similar reasoning may be done for the case when \( r_1 = r_2 \).

When the roots are complex conjugate \( r_1 = r + j\omega, r_2 = r - j\omega \), then the solution is given by:
\[ x(t) = \gamma e^{r(t-t_{2i})} \cos(\omega(t-t_{2i}) + \varphi) + \bar{x}_d \]
(7.70)

with \( \gamma = -\frac{\bar{x}_d}{\cos \varphi} \) and \( \tan \varphi = \frac{\varepsilon + \bar{x}_d r}{\bar{x}_d \omega} \). Now we get
\[ \dot{x}(t_{2i+1}) = \gamma e^{r \Delta_{2i+1}} \sqrt{\omega^2 + \omega^2 \cos(\omega \Delta_{2i+1} + \varphi + \Phi)} \]
(7.71)

with \( \tan \Phi = \frac{\omega}{r} \). Using the same arguments as in the real roots case, one sees that for \( \dot{x}(t_{2i+1}) \) to be arbitrarily small, we must have \( \cos(\omega \Delta_{2i+1} + \varphi + \Phi) \) arbitrarily small, from which we deduce that \( \omega \Delta_{2i+1} + \varphi + \Phi \) is arbitrarily close to \( \frac{\pi}{2} \). Now for \( \varepsilon \) arbitrarily small, \( \tan \varphi \to \frac{\varepsilon}{\omega} \), and \( \tan(\varphi + \Phi) \to +\infty \). But since \( \Delta_{2i+1} \) is assumed to be bounded away from zero (and strictly positive by definition), \( \tan\left(\frac{\pi}{2} - \omega \Delta_{2i+1}\right) \) is clearly bounded. Thus by contradiction we deduce that \{\( \Delta_{2i+1} \)\} converges to zero.

Now exactly the same reasoning may be done for the case of non-contact phases. It follows that if the velocities at times \( t_i \) converge towards zero, so do the intervals \( \Delta_i \). Since again the sequence \{\( t_i \)\} is infinite, if its limit is infinite also then \( (0, 0) \) is an equilibrium point of the system in (7.48). Clearly this is not the case, except if \( x_d = 0 \) (For the sake of brevity this case is not analyzed here; the analysis can be done using similar arguments). In conclusion, we have proved that the sequence \{\( t_i \)\} is either finite, or possesses a finite accumulation point. In both cases, we deduce that the equilibrium point of the system in (7.48) is asymptotically attained.
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relationship with the case of a rigid environment

In subsections 7.2.2 and 7.2.3, we have seen that some stability analysis may not be suitable in the sense that the conditions deduced on the feedback gains are obviously useless in practice as soon as the stiffness becomes too large. This means that if the designer wants to tune the feedback gains taking into account the unilateral feature of the constraint, and if he (or she) uses these analysis, he (or she) will necessarily not use them finally. However the last analysis proves that the equilibrium point of the system in (7.48) will be attained for any (strictly positive) value of the feedback gains, and any (bounded but arbitrarily large) value of the stiffness $k$. In the rigid limit case, the system is described by the equations in (7.49). Notice that this is exactly example 2.1 with a nonzero external force (the control input $u$) and with a restitution coefficient equal to 1. We cannot use here the results in problems 2.1 or 3.1, because there is a dissipation during the flight times.

Note also that by considering the elastic dynamical problem 7.49 we get as long as contact is maintained $x = \dot{x} = 0$, so that $V_c = \frac{1}{2}kx^2 = 0$ since the elastic potential energy vanishes. The only things that are modified in the rigid case are that since the intervals $[t_{2i}, t_{2i+1}] \rightarrow \{t_{2i}\}$, the distinction between instants $t_{2i}$ and $t_{2i+1}$ becomes worthless (we can take the notation $t_{2i} = t_k$). Consequently $\mu_{2i+1} = \beta_{2i+1} = 0$ while $\mu_{2i} > 0$ and $\beta_{2i} > 0$. Based on the analytical tools we have outlined one can easily adapt the foregoing analysis when $k = +\infty$. If $N < +\infty$, then necessarily after a finite time $x = \dot{x} \equiv 0$. Let us notice that as long as the restitution coefficient $\epsilon > 0$, then $\{t_k\}$ is an infinite sequence except if $\dot{x}(0^-) = 0$ and $x(0) = 0$. If $N = +\infty$, $T(t_k)$ converges as $i \rightarrow +\infty$, hence $\dot{x}(t_k) \rightarrow 0$. Note that only the flow of the first equation in (7.48) has to be considered in the reasoning. The remaining arguments are the same. One sees that the stability analysis is simpler in the rigid case, due to the fact that the contact equations are algebraic, no longer dynamical. This enables us to study the variation of $V_{nc}$ not only on smooth dynamics but also "during" contact, i.e. at impact times, i.e. in fact for all $t \geq 0$. Hence only $V_{nc}$ is needed to study the limit rigid case, since the contact phases reduce to instants $t_k$ and with $\sigma V_c(t_k) = 0$. It is finally note worthy that since $V_{nc}(t) \leq 0$ on $(t_k, t_{k+1})$ and $\sigma V_{nc}(t_k) = T(t_k) = 0$, one gets $V_{nc}(t_k^+) \leq V_{nc}(t_k^+)$ for all $k \geq 0$. Now since the restriction of $V_{nc}$ to the surface $\Sigma = \{(x, \dot{x}) : x = 0\}$, is such that $V_{nc,\Sigma}(t_k) = V_{nc}(t_k)$, one deduces that $V_{nc,\Sigma}$ is a Lyapunov function for the impact Poincaré map $P_\Sigma$. Therefore the stability analysis made in subsection 7.2.4 yields in the limit as $k \rightarrow +\infty$ Lyapunov stability of the map $P_\Sigma$.

We thus have proved the following:

Claim 7.4 Consider the closed-loop equations in (7.48). Then for any $\lambda_1 > 0$, $\lambda_2 > 0$, $k \in [0, +\infty]$, and for all initial conditions $x(0)$, $\dot{x}(0)$, $x \rightarrow \frac{4\lambda_2}{\lambda_1 + k}$ and $\dot{x} \rightarrow 0$ as $t \rightarrow +\infty$.

---

26 This is not the case for the compliant model since $V_{nc} = -\lambda_2 \dot{x}^2 - kxx$ during contact phases and $V_c = -\lambda_2 \dot{x}^2 + kxx$ during free-motion phases.
Remark 7.20 A distinction has to be made between two different cases of analysis: we may consider i) Either an arbitrarily large but bounded \( k \), ii) or a \( k \) that tends to infinity (that is implicitly a sequence of stiffnesses \( k_n \) with unbounded limit together with the corresponding dynamics). Clearly claim 7.4 can be concluded from the analysis in subsections 7.2.2 and 7.2.3 in case i), but not in case ii). The utility of the analysis proposed in subsection 7.2.4 is to enable us to draw conclusions in both the compliant and the rigid environment cases within a unique framework.

Remark 7.21 Let us note that the results on stability of hybrid dynamical systems in [426] theorem 1 do not seem easy to apply in our case, with the choice of the positive definite functions we have done.

Remark 7.22 We have been able to deduce the Lyapunov stability of \( P_{\mathbb{E}} \) from the study of the variation of two functions, \( V \) in (7.51) (with \( \tilde{x} = x \)), and \( V_{nc} \) in (7.65). Notice however that on \((t_k, t_{k+1})\), \( V \) is not guaranteed to be negative, except if \( x_d = 0 \) (although \( V(t_{k+1}^+) \leq V(t_k^+) \)). But we shall need in general for stabilization on \( \Sigma \) a strictly positive \( x_d \) (we could have denoted the term \( \lambda_1 x_d \) as \( F_d \) from the beginning, to make it clear that this is the value for the interaction force during permanent contact). On the other hand \( V_{nc} \) is only semi-negative definite on \((t_k, t_{k+1})\). Hence it will not be suitable for the application of the stability concepts developed either in [27] or in chapter 8, lemmas 8.1 and 8.2. We therefore face a dilemma: design a controller \( u_t \) such that there is a Lyapunov-like function which is negative-definite during the flight-times, and such that the collisions exist to assure stabilization on the constraint. As we shall see in chapter 8, this is not really necessary when the constraint surface is of codimension one. But for \( n \)-degree-of-freedom manipulators subject to a constraint of codimension \( \geq 2 \), multiple impacts will create difficulties. Then such a controller would be quite useful.

It would be nice to extend such analysis to more complex cases, for instance when an integral feedback is present in the control loop. The dynamical equations in (7.48) become:

\[
\begin{align*}
\begin{cases}
    m\ddot{x} + \lambda_2 \dot{x} + \lambda_1 x + \lambda_3 \int_0^x x(\tau)d\tau = \lambda_1 x_d & \text{if } x < 0 \\
    m\ddot{x} + \lambda_2 \dot{x} + (\lambda_1 + k)x + \lambda_3 \int_0^x x(\tau)d\tau = \lambda_1 x_d & \text{if } x \geq 0
\end{cases}
\end{align*}
\]

(7.72)

and the rigid limit case in (7.49) is extended similarly. In fact we have seen that the stability analysis led in subsection 7.2.4 is closely related to the Lyapunov stability of the Poincaré map of the rigid case, with Poincaré section \( \Sigma = \{(z, x, \dot{x}) : x = 0\} \) with \( z = \int_0^x x(\tau)d\tau \). As we shall see in chapter 8, subsections 8.5.2 and 8.5.3, such a stability analysis is not easy to do.

Notice that if \( P_{\mathbb{E}} \) is Lyapunov stable, then since the solutions of the compliant problem converge to those of the rigid one, the corresponding transition phase for the compliant problem must be stable for \( k > k^* \), \( k^* < +\infty \). This means that one is
able to deduce the stability of a rebound phase with a compliant environment from that of the Poincaré map of the rigid body case. We could have used this result to prove rapidly that for large enough stiffness $k$ in (7.48), $(0, 0)$ is a Lyapunov stable equilibrium point of the system in (7.48), for any $\lambda_1 > 0$, $\lambda_2 > 0$, $x_d > 0$. Whether such a "converse" proof may be extended to more complex cases as in (7.72) remains an open question. Indeed showing the stability of the impact Poincaré map is not easy in general. But there are important cases for which one is able to prove the local stability of the impact Poincaré map without explicit knowledge of the trajectories: we saw in subsection 7.1.4 that it is possible in certain cases to guarantee the existence of periodic trajectories, and to calculate the corresponding Jacobian of the impact Poincaré map to deduce \textit{via} Floquet's method their stability. If at the same time one is able to prove convergence of the solutions of approximating problems $P_n$ towards those of the rigid body problem $P$, then one may conclude about the behaviour of solutions of $P_n$ for $n$ large enough. This is related to results on persistence of closed orbits for $C^1$ vector fields [200] chapter 16, §2, theorem 1: if the Poincaré map Jacobian of a periodic trajectory of a vector field $f(x)$ has no eigenvalues equal to 1, then any vector field $g(x)$ close enough to $f(x)$ also possesses a periodic trajectory.

We retrieve here the fact that rigid body dynamics can yield simplified dynamics which may be used to deduce the behaviour of an "almost-rigid-body" system which would otherwise be hard to study. This confirms once again the fact that rigid and compliant models do not have to be opposed, but are rather complementary.
Chapter 8

Feedback control

In the foregoing chapters we have described the dynamics of mechanical systems, whose position is constrained to move in a certain domain of the configuration space. We have investigated those systems from several points of view, starting from mathematical considerations (nature, existence, uniqueness of solutions, approximation of rigid problems via sequences of compliant problems). Then we have studied the dynamical equations, which include restitution rules at the shock instants. We have seen that several different formulations can be used to describe the same problem. They are not all equivalent. In particular we have proposed a general framework for investigating impact dynamics, through the use of the kinetic metric. Throughout the chapters, we have discussed the advantages and drawbacks of rigid and compliant models. We have seen that some rigid problems may be undetermined, a fact that is currently met when perfect rigidity is assumed (think of the chair with four legs: it is impossible to calculate the reactions at each leg without introducing some compliance). However rigidity allows most of the time to simplify the dynamics, an important point for engineers who have to do calculations. Moreover replacing rigid bodies by compliant ones may not always be a good solution, especially for numerical investigations. On the other hand, compliance may in certain cases be a more accurate model. Finally, we have studied the problem of stability of trajectories of measure differential equations. We have pointed out the limitations of the available studies, and how some more general stability results can be obtained, by considering the solutions not as piecewise continuous functions, but as functions of bounded variation.

This chapter is devoted to study the control of mechanical systems submitted to unilateral constraints. By control we mean that one is able to define inputs and outputs for the system, and that the inputs may be chosen as feedback laws, to drive the output towards a desired target. This is the general goal of systems control theory, see e.g. [554]. We begin by defining some general facts that may have to be verified by an impacting system. Then we describe within this setting different systems where the percussions play a major role. The major part of the chapter is devoted to present a stability analysis framework that is suitable for
feedback controlled of a class of impacting systems of rigid bodies. The principal
difficulties posed by the closed-loop stabilization are highlighted, and the remaining
open problems are pointed out.

8.1 Impacting robotic systems

By impacting robotic systems we mean devices involving robot manipulators that
collide with some other mechanical system. As we have seen, such systems can
always be written in the general form of Lagrangian system submitted to some uni-
lateral constraints. Roughly, it suffices to define the distance between the different
bodies as being the constraint. What are the general requirements that a designer
would like to impose to such a system? Let us consider the following ones

- 1) a) The (smooth) trajectories remain bounded between the impacts (we
mean trajectories in the state space), and/or b) the system tracks desired
trajectories between the impacts.
- 2) The impact sequence \( \{t_k\} \) exists, i.e. there is at least one impact. It may
have to be either a) finite or b) infinite.
- 3) The limit of the sequence \( \{t_k\} \) may be either a) finite or b) infinite.
- 4) The impact times may be either a) separated, \( t_{k+1} < t_k \), or b) possess finite
accumulation points, or c) an infinite accumulation point.
- 5) The velocity at \( t_k \), \( \dot{q}(t_k^+) \), tends towards a desired value \( \dot{q}_d \).
- 6) The position at \( t_k \), \( q(t_k) \), tends towards a desired value \( q_d \).

Note that 2) 3) and 4) are necessary to describe the form of the sequence \( \{t_k\} \).
Indeed 2a) implies 3a), but not 4a). 2b) may be with 3a) (hence 4b)), or with
3b) (hence 4a) or 4c)).

Let us now consider the following tasks (This list is not exhaustive. Some other
robotic tasks might certainly be defined).

- Juggling

The goal of juggling is to make one or several objects, submitted to gravity
acceleration during flight phases and to impacts with the robot, track some
trajectories just using the impacts. The bouncing ball on a vibrating table
constitutes the simplest juggler. Note that 1) a) cannot be a priori assumed,
but must be guaranteed via some control. Indeed, the following is true (it
is assumed that the table position is given by a periodic analytic function
\( f(t) = f(t + T) \), and that the table has infinite mass, hence represents a
unilateral constraint for the ball)
Theorem 8.1 (Pustylnikov [447]) Assume that there is an integer \( N > 0 \) and a \( t_0 \) such that \( \dot{f}(t_0) = \frac{TgN}{2} \), \( -g < \dot{f}(t_0) < 0 \). Assume also that \( \dot{f}(t_0) \neq -\frac{g}{2} + \frac{g}{2} \cos \left( \frac{2\pi m}{n} \right), m = 0, \pm 1, \ldots, \pm n, n = 1, 2, \ldots, 262 \), and that there exist two functions \( a_0(\cdot) \) and \( a_1(\cdot) \) such that

\[
a_0(\dot{f}(t_0)) \leq t(t_0) + a_1(\ddot{f}(t_0)) \leq 0,
\]

with \( a_0(\dot{f}(t_0)) \neq 0 \). Then there exists in the plane \((t, \dot{t})\) a set of positive measure of initial data such that the velocity after the impacts tends to infinity \((\ddot{t})\).

This result is true for purely elastic impacts, hence it may lack from practical importance \((^2)\). Also it seems possible to obtain an infinite velocity because the table mass is assumed to be infinite. Otherwise this would mean that with a bounded input (that creates the table motion) one could supply an infinite energy to the ball. Clearly as long as the table mass is infinite, then one cannot use any kinetic energy boundedness arguments, and this renders such a surprising result possible. However it proves that if the ball and the table are very rigid and with a restitution close to 1, then some motion of the robot will make the contact velocity become very large, and consequently the percussion and the height attained by the ball. Let us recall that if the restitution coefficient is \( < 1 \), then velocity is bounded, see chapter 7, section 7.1.4. 2b) and 3b) must be guaranteed \textit{via} feedback control in closed-loop. In fact the goal may be to create a periodic motion \((\text{a limit-cycle, or a fixed-point of the associated Poincaré map})\), but this is not necessary. The desired motion may be nonperiodic. 4a) is likely to be verified if the closed-loop trajectory is a limit-cycle, i.e. the impacts occur periodically. 5) does not seem \textit{a priori} necessary, although it may be a consequence of the control strategy. 6) means that the robot and the object collide at certain desired places, which may be required, but is not necessary.

- **Catching**

Catching and juggling tasks are quite close one to each other. It can be argued that catching is a special case of juggling. We however keep the separation between them, and consider that catching means to obtain a permanent contact between the robot and the object in finite time. 1) a) and 2) have to be guaranteed \textit{via} feedback control. 3a) is required, since we want to catch in finite time. 4b) will occur if for instance a model with restitution \( > 0 \) is used. 5) must hold with \( \dot{q}_d = 0 \). 6) is not necessary, the catching may occur at an indefinite place.

- **Stabilization of a manipulator against an obstacle**

The goal may be to study the transition phase between a free-motion and a constrained motion phases. Concerning the transition phase alone: 1) a) and

\(^1\)In other words, the trajectories of the impact Poincaré map increase in velocity to infinity.

\(^2\)Note anyway that as pointed out in [34], if the case \( e = 1 \) did not exist in nature, then all molecular motion would long since have ceased. But we leave here engineering.
b must be guaranteed via suitable feedback, as well as 2b) (if \( e > 0 \)). 3a) is a consequence of the impact model, as well as 4b). 5 is naturally with zero desired value for the normal components, whereas 6) is fixed by the constraint (after stabilization, then the robot evolves on the smooth submanifold defined by the constraint).

As we shall see, things are much less simple when one considers a complete task that we define below, because the requirements are then more complex (track position during free-motion phase, track interaction desired force during constrained phase), and the whole closed-loop system itself is much more complex.

- **bipedal locomotion**

The goal here is to make a bipedal mechanism walk on a ground, with actuators at some joints. 1): the trajectories of the bodies must not only be bounded, but track some path (although this is not obvious: this should be a consequence of a definition of what walking exactly is, a question whose answer is not so simple). The main goal may be to create a periodic motion of the gravity center with one impact per period. 2) is easy to obtain since the system is under the influence of gravity. 2a) or 2b) depend on the task (It seems reasonable to suppose a finite length walk), and on the model of impact. If \( e = 0 \), then 3a and 4a are verified. 5) and 6): similarly to the robot stabilization case, some components of these values will be given by the constraints, the rest depends on the closed-loop dynamics (consequently on the control law). This topic has received attention in many papers, see e.g. [98] [100] [101] [192] [217] [218] [219] [221] [364] [266] [277] [450] [462] [465] [550] [551] [563] [611].

- **Possible other tasks**

Let us consider for instance a controlled hammer that strikes a nail. Then the control law must guarantee 1). Intuitively, a periodic trajectory such that the hammer collides the nail at repeated instants \( t_{k+1} = t_k + T, T > 0 \), is suitable. If \( e = 0 \), there will be one impact per period. If \( e > 0 \), there will be rebounds phases that are likely not to be desired in that kind of device. The input should somewhat reduce their effects by imposing enough dissipation during flight times. This means that roughly, one must separate the task in approach phases to strike with a certain impulse magnitude (hence a desired velocity 5), but 6) is given by the nail position), and rebounds phases that should be as short as possible to guarantee 3a) and 4b), if 2a) and 4a) cannot be avoided. Closely related to juggling is robotic tennis-table players [8].

Let us finally mention that the study of control and dynamical behaviour of mechanisms with clearances should incorporate not only backlash with hysteresis but impact models as well. For instance adaptive control of systems with backlash and hyteresis has been investigated in [530] [531] [532]. Basically, and disregarding the adaptive control action, the controllers in [530] [531] [532] are such that the system switches instantaneously from one side of
the clearance to the other side, when the two bodies are separating. Hence
the controller guarantees that the system is never ”inside” the clearance, but
rather always lies on the constraints. It is argued that the switching effect can
be smoothed by replacing the ideal discontinuous controller by an approxima-
tion (which is likely to work since, at least in the known parameters ideal case,
Lyapunov stability of the closed-loop system is shown with a Lyapunov func-
tion, thus similar results as in [114] can apply). But the collision phenomena
have not been taken into account in [530] [531] [532]. This has crucial con-
sequences on the analysis and on the actual system behaviour (and we hope
the reader is at this stage convinced of this fact!). In particular we saw in
chapter 7 that the dynamics of the impact damper are quite complex, and are
in fact far from being modeled by an hysteresis. One may try to study the
conditions such that the impact damper model is as close as possible to the
backlash model in [530] [531] [532] (for instance $\varepsilon = 0$, the mass ratio $\frac{m}{M}$
very small so that after an impact the velocity of $M$ remains very small), and try
to adapt their controllers to that case, or study its robustness with respect to
shock dynamics. From a general point of view, two directions should be fol-
lowed for studying control of systems with clearances: the study of robustness
properties of controllers designed for the ideal case (zero clearance), and the
design of controllers that take explicitely the backlash and the collision models
into account. The studies on the dynamics of the impact damper (see chapter
7) may be of some usefulness in this setting for an extension/modification of
Tao and Kokotovie’s controllers in [530] [531] [532].

Various possible dynamics of systems subject to unilateral constraints are depicted
in figure 8.1.

**Remark 8.1** Consider the problem of the lamina falling on a rigid and possibly
moving obstacle. Then if we consider that the control input is the motion of the
obstacle, we retrieve the juggling task. If we consider that input is the torque
and force applied on the lamina, then we retrieve the stabilization of a robot (the
lamina may be thought of as the end-effector of the robot). Those considerations are
important, because by changing the control, one modifies drastically the properties
of the open-loop system, like controllability. It is an open problem to extend the
work in [574] to such study: how does the change of input modify the controllability
properties of the associated discrete Poincaré map?

**Remark 8.2** These systems belong to what is sometimes called *vibro-impact* sys-
tems. They have received much attention in the literature. For the moment, the
results from those investigations, essentially centered around complex dynamics,
chaotic behaviours and bifurcations at grazing trajectories, have not really been
used in the setting of impacting robotic tasks. Some works in this direction can be

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3So-called $C$-bifurcations, see e.g. [80] [106] [81] [146] [147] [148] [149] [236] [301] [401] [402]
[237].
CHAPTER 8. FEEDBACK CONTROL

complete robotic task stabilization of a limit cycle or on the constraint (juggling, catching)
multiple impact (finite multiplicity)

stabilization of a periodic trajectory (2 impacts per period): impact damper, clearances, backlash...
motion around the singularity (bipedal walking robot, rocking block)
multiple impact (infinite multiplicity) (rebounding block)

Figure 8.1: Impacting robotic systems.
8.2. CONTROL OF COMPLETE ROBOTIC TASKS

found in [83] [82] [573] [172] [458] [620]. Recently Vincent [555] [556] used the chaotic dynamics of the bouncing ball on a vibrating table to stabilize a juggling task. Concerning the control of bipedal robots, it seems that the first attempt to use Poincaré maps and Floquet's theory (4) to study stability for walking machines are the works in [218] [219] [98]. This can be seen as the extension to (closed-loop) bipedal manipulators, of the stability techniques described in chapter 7, section 7.1.4 applied to simple systems such as the impact oscillator (with one or two constraints).

In the next part, we focus on the control of robot manipulators during general tasks.

8.2 Control of complete robotic tasks

In most of the robotic tasks, the manipulator has to move alternatively in spaces free of obstacles and along certain surfaces: this is what we shall call complete tasks. There correspond free-motion phases (no constraints) and constrained-motion phases (bilateral constraints), that have been separately thoroughly studied in the robots control literature (via motion control or hybrid force/position control respectively). When the end-effector attains the surface(s) of constraint, impact phenomena occur in a so-called transition phase. It is thus necessary to include the percussions effects in the control design and in the closed-loop stability analysis. Roughly, the available works in the literature can be split into two parts: experimental studies on the control of the transition phase for one degree-of-freedom systems, and stabilization of $n$-degree-of-freedom manipulators with $m$ unilateral constraints during complete tasks.

8.2.1 Experimental control of the transition phase

In this subsection we review simple control strategies that have been proposed in the literature to improve the transition phase behaviour of a manipulator colliding an obstacle. The works described below, far from solving the general problem of controlling mechanical systems subject to unilateral constraints, are for the moment restricted to 1-degree-of-freedom robots colliding with either rigid or compliant environments. They are mainly experimental.

In [207] an experimental evaluation of impedance control during a contact task is presented; the strategy proves to behave stably during the transition phase, even if the environment is very rigid. It is clearly one of the simplest solution one can imagine to implement, both for the control law itself and the few assumptions on the environment it requires. However the main drawback of the method is that

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4That mainly consists of a linearization of the dynamics around some periodic trajectory, and study of the eigenvalues of the Jacobian to deduce which type of bifurcation occurs when a parameter is varied [227] chapter 7.
as the same feedback gains are used during the whole task, the position tracking performances are poor. In [602], experiments on a 1-degree-of-freedom robot are reported. An integral force controller that acts as a low-pass filter for high frequency components of impact transients plus velocity feedback that damps the system is shown to perform well during the transition phase with the following strategy: i) Approach with constant velocity while monitoring sensed force for impact detection, ii) Turn on force regulation on the desired steady state force magnitude, iii) Leave the controller unchanged whatever happens after the first impact has occurred. The theoretical analysis and the experimental results are in close agreement. A detailed presentation of the experiments is given in [602]. Surprisingly enough, these results are contradicted in [560] who show that integral force control behaves at best with oscillations during the transition phase: due to the integrator wind-up, the second impact has larger magnitude than the first one. Moreover too large integral gains lead to instability. The discrepancy between the results in [602] and [560] essentially comes from the fact that the environment is either rigid [602] or compliant [560]

**Remark 8.3** When facing a control problem, one has to consider three properties of the closed-loop solutions: existence (i.e. nature), uniqueness, and stability. The first two steps are necessary to get a wellposed problem. The third step has crucial practical consequences, and cannot be treated without the previous ones being clarified. As we saw above, existence and uniqueness for the rigid case are often analyzed through sequences of compliant problems with known and strong properties. What about stability? It is worth noting that the problem we just discussed constitutes an example where the closed-loop trajectories of smooth problems are unstable, but stability holds for the rigid limit problem. It has also been noticed in [153] for the permanent contact case that stability of integral force feedback depends on the stiffness value, and in some sense increases with the stiffness. It may also happen that some sort of stability that holds for the smooth problems - e.g. quadratic Lyapunov stability - is not true at the limit for bounded feedback gains, where only boundedness and asymptotic convergence holds: we have studied this problem in section 7.2 for the case of a 1-degree-of-freedom system striking an elastic environment. These results show that it is not in general possible to carry stability results from the compliant to the rigid case, and vice-versa. An important question for control purposes is about robustness of controllers designed from a rigid model, when some flexibility is present in the environment or the manipulator's tip, see remark 8.33. Notice also the following: to the best of our knowledge, global stability of the impact transition phase when a proportional+integral force feedback is used (that is experimentally shown to behave well in [602]) has not yet been shown. Using the results in [574], local stability results only can be drawn. This is related to the difficulty in general to get global analytical results for apparently simple systems with algebraic impact conditions.

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The work in [602] contains the most advanced study to date, since the simulations are based on a rigid impact model and compared with experiences. But no
theoretical stability analysis is done, and note that the study concerns the impact phase only, not a whole robotic task. In [560], several control strategies are tested on a one degree of freedom robot. In particular, it appears that integral force control leads to a poorly behaved transition phase, whereas proportional gain force feedback behaves correctly. However integral action is needed when contact is established to get good force regulation. (It is also shown that impedance control and proportional force feedback are equivalent when the environment is very stiff). The main conclusion in [560] is that three distinct controllers have to be used: position control, impact control and proportional gain force feedback. However no general stability analysis is given to corroborate these essentially experimental results. In [612] the effects of impacts in the joints of a rigid manipulator are studied, using a Newton-Euler recursive algorithm to describe the dynamics. The authors of [2] also investigated the collisions effects in n-degree-of-freedom manipulators. They studied the distribution of kinetic energy after the impact in the robot links. It is also apparent in the results of [2] that both the normal and the tangential velocity of the contact point between the robot’s tip and the obstacle undergo discontinuities. This is easily understandable from the developments in chapter 6: we saw that starting from a generalized Newton’s restitution rule, then due to inertial effects and even without any friction, the tangential velocity of the contact point possesses in general a jump.

In [268], the authors propose to increase damping (velocity feedback) during a transitory stage after the first impact has occurred. This so-called impact transition control stage [268] aims at dissipating the impact energy to avoid excessive bouncing, and minimizes force overshoot at the moment of impact. Experimental results on a one axis impact testbed are presented in [226]: several control strategies available in the literature are implemented and shown to improve the transition phase behaviour. Experiments have also been led in [491] in the framework of multifingered hands. Collisions of fingers of different nature with a rigid obstacle are tested. The results show how the peak impact forces vary with the materials in contact.

Purely kinematic solutions to reduce impact effects are studied in [168] [564] [613] [313], mainly using the relationship between the impulsive force and the generalized velocities jump, see example 1.3. Possible redundancy of the manipulator provides more freedom to reduce the impact magnitude [564] [313], noting that anyway, the relationship derived in example 1.3 remains true in the redundant case (The Jacobian is no longer square in that case). Recently the impacts effects in space manipulators has been investigated [116] [603] [604] [605]. Impact devices have been built to estimate the collision effects. They are composed of piezo-electric sensor to measure the impact force and encoder to measure pre and postimpact velocities. The improvement of the impact control via the addition of compliance at the contact has been studied in [425] [267] [423]. Notice that it is evident that if the stiffness of the bodies that collide decreases, the interaction forces decrease as well. However we reiterate that in some mechanisms, rigid and strong impacts may be desired (think of a hammer-like task where the robot has to strike a nail). On the other hand compliance may not be desired since it decreases the displacements
accuracy, and can introduce difficulties for the free-motion control. The maximum force during an impact process between two flexible bodies and using a simple linear model as in chapter 2, section 6.3 is calculated in [253]. Comparisons are made with experimental results using various materials.

8.2.2 The general control problem

The last fifteen years have witnessed a considerable interest on the problem of control of mechanical systems. Both motion control and hybrid force/position control cases have received much attention. In the first case, the system is assumed to evolve in a space free of obstacles, and is described by a set of ordinary differential equations (ODE). In other words, and since there always exist obstacles, one assumes that the system evolves in Int(\Phi) for all \( t \geq 0 \). Feedback linearization as well as more specific controllers (adaptive, robust control) have been proposed [501] [508]. In the second case, the system is assumed to evolve on a constraint submanifold of the form \( f(q) = 0 \), i.e. on \( \partial \Phi \). Interaction forces between the manipulator's tip and the obstacle have to be taken into account in the analysis. Solutions based on a decoupling between free tangential motion along the surface \( f(q) = 0 \) and the constraint normal direction have been proposed [324] [606] [607]. It is worth noting that in this latter case, the constraint is assumed to be verified for all times, without any consideration of the possible transition between configurations \( q \) such that \( f(q) > 0 \) and configurations such that \( f(q) = 0 \). This has considerable consequences, since the system's trajectories can still be considered as smooth time functions. Hence all available existence, uniqueness and stability of solutions results can be applied directly to such systems.

Modeling of collisions in kinematic chains, see section 5.2 and control of complete robotic tasks, i.e. tasks involving free motion as well as constrained motion phases (like for instance deburring and grinding operations) are closely related (In fact, as always in feedback control, one needs models before designing a controller!). A lot of experimental works have been devoted to study the transition phase control, which occurs when the robot's tip strikes the environment's surface, see subsection 8.2.1. In particular the results in [560] (flexible environment) and [602] (rigid environment) show that the environment's stiffness has a significant influence on the transition phase behaviour. Pierrot et al [437] show that rigid body models can provide in certain cases better prediction than flexible ones, using appropriate nonsmooth mathematical tools [383]. Experiments have been presented in [546] on a Puma robot that confirm the usefulness of nonsmooth dynamics in the modeling of robot collisions. Other investigations on rigid body impacts on n-degree-of-freedom manipulators may be found in [2]. Pioneering fundamental work in the field of complete robotic tasks control can be found in Mills and Lokhorst [360], who considered control of n-degree-of-freedom rigid manipulators evolving either in free-space or in contact with compliant environment. A simple switching controller is considered which basically consists of a motion control law and a force/position control law applied when contact is established or not. Roughly speaking, the open loop as well as
closed-loop systems are composed of two smooth vector-fields. The stability analysis consists of application of Lyapunov stability concepts for simple hybrid dynamical systems \cite{64}, i.e. one requires a unique positive-definite function $V$ such that $V$ is negative definite along both vector-fields trajectories. This is attained via suitable feedback gains choices. Existence of solutions at the switching times is carefully analysed in \cite{360} via concepts related to differential inclusions (since at that times the control input is not uniquely defined). Other related works can be found in \cite{341} \cite{87} \cite{341} \cite{341} \cite{341} \cite{341} \cite{341} \cite{341} \cite{341}. It is however note worthy that the systems considered in \cite{360} do not belong to the class of systems subject to unilateral constraints, because of the environment’s compliance. Therefore this result cannot be considered as an extension of the work on constrained manipulators \cite{324}, i.e. control of systems represented by the following set of equations

\begin{align}
M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) &= u \quad (8.1) \\
f(q) &\geq 0 \quad (8.2) \\
\text{Law (percussion, velocity)} &= \text{true} \quad (8.3)
\end{align}

where (8.1) is the dynamical equation of rigid manipulators in free-space: $q \in \mathbb{R}^n$ is a vector of generalized coordinates, $M(q) \in \mathbb{R}^{n \times n}$ is the positive-definite inertia matrix, $C(q, \dot{q})\dot{q}$ contains Coriolis and centrifugal acceleration terms, $g(q) \in \mathbb{R}^{n \times 1}$ is the vector of generalized gravity torques, $u \in \mathbb{R}^{n \times 1}$ is the control input vector. (8.2) represents the set of unilateral constraints, i.e. the subspace $\Phi \subset \mathbb{R}^n$ within which the system evolves. (8.3) is a physical law that relates the interaction between the robot’s tip and the surface $f(q) = 0$, and the generalized velocity $\dot{q}$, when contact is made at $t = t_k$, i.e. $f(q(t_k - \delta)) > 0$ for any small enough $\delta > 0$ and $f(q(t_k)) = 0$.

More generally, (8.1) may represent the dynamics of any Lagrangian system provided one knows stabilizing controllers for free-motion and constrained-motion phases, as we shall see later. This may be the case for instance for flexible joint robots constrained by a fixed environment. This may also include manipulators with dynamical environments, i.e. environments which possess a dynamical behaviour. Then the equation in (8.1) represents the dynamics of the whole system, and the constraint $f(q)$ represents some distance between both systems. In general such systems as well as mechanisms with clearances and backlash will not fall into the class of systems we consider in the following. Juggling robots will neither. Indeed such systems include variables which evolve freely between impacts, like the ball in a juggling robot, see remark 7.13 and (7.45) (7.46), or the dynamic environment if it is passive. Their control is therefore, perhaps not more difficult, but surely different. We may represent systems as in (8.1)-(8.3) as shown in the block diagram in figure 8.2. Contrarily to the juggling robots case (see 7.3), we do not have this time to relate one part of the dynamics (uncontrolled) to the (controlled) one via the restitution rule. The collisions are to be considered here more as disturbances that the bounded controller would indirectly control between them.

In this section we shall consider the control of systems as in (8.1) (8.2) (8.3), with $f(q) \in \mathbb{R}^m$, i.e. we model the contact process from a rigid body point of
view. Before going on with the control problem, let us summarize some basic facts about dynamics with unilateral constraints that we described throughout the foregoing chapters. We restrict ourselves to codimension one surfaces of constraint, or to hypersurfaces $f_i(q) = 0$, $i \in \mathcal{I}$, which are mutually orthogonal with respect to the kinetic metric defined as $<x,y> = x^T M(q)y$, $x,y \in \mathbb{R}^n$, see chapter 6. Here $\mathcal{I} \subset \{1, \cdots, m\}$ represents the set of constraints which are attained at the same time, through a multiple collision (either the singularity is reached directly, or through successive collisions with the hypersurfaces). We assume that $f_i(q) = 0$, $i = 1, \cdots, m$ are frictionless. The reasons for these assumptions follow from the developments in chapters 2, 5 and are recalled now:

- First we are interested in investigating the motion of a Lagrangian mechanical system evolving in a domain $\Phi$ of its configuration space. The boundary of such domain, $\partial \Phi$, is an hyperspace in $\mathbb{R}^n$. We may assume that $\partial \Phi$ is smooth, and that it can be represented analytically as $f(q) = 0$, for some smooth function $f : \mathbb{R}^n \rightarrow \mathbb{R}$.

- Second, it might be that $\partial \Phi$ possesses some singularities, given for example by the intersection of 2 hypersurfaces $f_1(q) = 0$ and $f_2(q) = 0$. In this case, one could admit the unilateral constraint in (2) to be represented as $f_1(q) \geq 0$, $f_2(q) \geq 0$. But the main problem we shall meet is to define the impact law in (8.3). It happens that if only one constraint is attained at a time $t = t_k$, then such impact rule is well-defined, see chapter 6. But if several hypersurfaces are attained simultaneously with a multiple collision, then it means that the system strikes the domain’s boundary $\partial \Phi$ at a singularity. At this singularity the normal direction is not well defined and the collision rule is not well-defined neither in general. The problems related to the definition of a restitution operator in this case have been reviewed and discussed in chapters 5 and 6. As we have seen, the definition of the normal direction is crucial for the impact rule (8.3) to be given a meaning. Hence a simplifying solution is to assume that the normal direction at the intersection of the 2 hypersurfaces is uniquely defined, i.e. we assume $\partial \Phi$ is smooth everywhere, i.e. we in fact restrict ourselves to codimension one surface $f(q) = 0$ to define the domain $\Phi$.

- Third, we saw that there is however an important case where single impacts

![Figure 8.2: Lagrangian system with unilateral constraints (block diagram).](image-url)
(m = 1) restitution rules can be extended to multiple impacts [232] [292] when several constraints \( f_i(q) = 0, \ i \in \mathcal{I}, \) are attained simultaneously: when such surfaces verify an orthogonality condition expressed as:
\[
\nabla_q f_i(q)^T M^{-1}(q) \nabla_q f_j(q) = 0 \tag{8.4}
\]

\( i, j \in \mathcal{I}, \ i \neq j. \) Then the dynamical algebraic equations at the shock instants can be given a coherent meaning, see subsection 6.5.8 and the introduction of section 6.5.

Moreover, existence and uniqueness of solutions for such dynamical systems is in fact a hard problem. It has been proved in some particular cases, limited to codimension one constraints or with \( T_L = 0, \) see chapter 2, theorem 2.1. In the following study we shall content ourselves with existence results, disregarding uniqueness problems. This choice is done mainly because of the great difficulty in proving such results for general impacting systems (and such studies are clearly outside the scope of this work), and also since there are no uniqueness results available in the mathematical literature for the dissipative shocks case which is of main interest for practical control purposes.

In summary, unlike the permanently constrained case where one supposes that the system evolves on a submanifold \( f(q) = 0 \) with \( f : \mathbb{R}^m \rightarrow \mathbb{R}^m, \ m \geq 1, \) things are not so simple when the transition from free to constrained motion is taken into account. Although we do not claim that the case of several unilateral constraints should be disregarded (it does on the contrary represent a challenge in impact dynamics to define impact rules for such cases as we saw in chapters 5 and 6) we prefer in the following to restrict ourselves to the above cases. The results were outlined in [74] [407], see also [77].

8.3 Dynamic model

In this section we provide some explanations and details concerning the dynamic model that will be used for control design purposes. We also give the definition of the trajectories using the result in [416], see chapter 2, section 2.2.

8.3.1 Impact dynamics

Codimension 1 constraint

Before writing down the whole set of equations we choose to model the system, let us summarize the basic facts concerning impact dynamics that we described in foregoing chapters. As long as the configuration \( q \in \mathbb{R}^n \) is such that \( f(q) > 0, \) then the system’s trajectories \( q(t), \dot{q}(t) \) are well-defined as absolutely continuous time functions, solutions of the ODE in (1) (where \( u \) is assumed to satisfy some classical regularity conditions). The same conclusions hold when \( f(q) \equiv 0 \) on a nonzero time
interval. Now assume that for some \( t = t_k \), \( f(q(t)) > 0 \) for \( t \in [t_k - \delta, t_k) \) and some \( \delta > 0 \) small enough, and \( f(q(t_k)) = 0 \).

Assume also that \( \dot{q}(t_k^-)^T \nabla_q f(q(t_k)) < 0 \) where \( \dot{q}(t_k^-) = \lim_{t \to t_k^-} \dot{q}(t) \) and \( \nabla_q f(q(t_k)) \) is the gradient of \( f(q) \) at \( t = t_k \) (that is supposed to be different from 0 in the region of interest). This condition means that the velocity points outwards the domain \( \Phi \) at the point \( q(t_k) \in \partial \Phi \). Then a collision at \( q(t_k) \) occurs and the right-limit of \( \dot{q}(t) \) at \( t_k \), denoted as \( \dot{q}(t_k^+) \), possesses a discontinuity such that \( \dot{q}(t_k^+)^T \nabla_q f(q(t_k)) \geq 0 \), i.e. the rigid velocity points inwards \( \Phi \). Recall that the jump in \( \dot{q} \), denoted as \( \sigma_{\dot{q}}(t_k) \), has to be specified through a so-called \textit{restitution rule} which relates post and preimpact velocities. We denote it for the moment as:

\[
\dot{q}(t_k^+) = R(\dot{q}(t_k^-)) \tag{8.5}
\]

**Remark 8.4** As we saw throughout the foregoing chapters on impact modeling, the fact that a jump in the velocity is necessary when the surface \( f(q) = 0 \) is attained in a non-tangential manner, and that the impact rule must be such that the postimpact velocity points inwards \( \Phi \), render the domain \( \Phi \) invariant under the dynamics. In viability theory language [24], \( \Phi \) is a \textit{globally viable domain}. The sweeping process formulation described in section 5.3 precisely aims at studying a general model that makes \( \Phi \) such a domain. In fact all the restitution rules are defined for this goal. Whether they fit with experimental results is another point.

At \( t_k \) the acceleration is given by

\[
\ddot{q} = \sigma_{\dot{q}}(t_k)\delta_{t_k} \tag{8.6}
\]

where \( \sigma_{\dot{q}}(t_k) = \dot{q}(t_k^+) - \dot{q}(t_k^-) \). It is then possible to prove (see section 1.1) that at \( t = t_k \) the dynamical equations become:

\[
M(q(t_k))\sigma_{\dot{q}}(t_k) = P_q \tag{8.7}
\]

It is also important to recall that \( q(t) \) is continuous at impact times, see section 1.1. We saw that equations (8.5) and (8.7) make the impact problem \textit{complete}, in the sense that given preimpact conditions \((q(t_k), \dot{q}(t_k^-))\) one is able to calculate both \( \dot{q}(t_k^+) \) (from (8.5)) and the percussion vector (from (8.7)).

Let us now recall the basic facts on the restitution rule in (8.5), which we developed in chapter 4. We saw (see chapter 4, claim 6.1) that this rule is given by:

\[
\dot{q}(t_k^+)^T \nabla_q f(q(t_k)) = -c q(t_k^-)^T \nabla_q f(q(t_k)) \tag{8.8}
\]

which can be seen as a \textit{generalized normal rule}. (Note that \( \dot{q}^T \nabla_q f(q) \) is a scalar, hence only one coefficient is needed in (8.8)). The remaining part of the velocity can be computed from (8.7). Indeed there are \((n + 1)\) unknown parameters to the problem (the \( n \) velocities components \( \dot{q}(t_k^+) \) and the unique component of \( P_q \) which verifies \( P_q = p_q \nabla_q f(q) \) for some \( p_q \in \mathbb{R}^n \) since the constraint surface is frictionless) and (8.7) together with (8.8) provides us with \((n+1)\) equations. We shall come back later on the calculations of postimpact velocities using some particular coordinates, such as the ones studied in [324].
8.3. DYNAMIC MODEL

Constraints of codimension $\geq 2$

In the case $f(q) \in \mathbb{R}^m$, $m \geq 2$, there may be several hypersurfaces attained at the same time, i.e. the system is submitted to a multiple collision, see chapter 6. As we saw, if $\Phi$ has a smooth boundary, then one may assume that $\partial \Phi$ is actually given by $f(q) = 0$, $f(q) \in \mathbb{R}$. Let us rather assume that at such singular points, $\partial \Phi$ is not smooth so that it is really given by the intersection of several hypersurfaces. The question is: can the restitution rules in (8.8) be generalized to such cases? Does this generalization yield a coherent result? The answer is yes, if the hypersurfaces that form the singularity verify (8.4), see chapter 6, and in particular section 6.5.8 and theorem 6.1. We saw that it is then possible to define a generalized velocity transformation ($^5$):

$$\begin{bmatrix}
\dot{q}_{\text{norm}} \\
\dot{q}_{\text{tang}}
\end{bmatrix} = \begin{bmatrix}
\eta_q^T \\
\eta_q^T
\end{bmatrix} M(q)\dot{q}$$

(8.9)

where the matrices $\eta_q \in \mathbb{R}^{2 \times n}$ and $\eta_q \in \mathbb{R}^{(n-2) \times n}$ are defined in chapter 6. $\dot{q}_{\text{norm}}$ and $\dot{q}_{\text{tang}}$ represent the coordinates in the frame $(n_q,1, n_q,2, t_q,1, \ldots, t_q,n-2)$ of the projections of $\dot{q}$ on $n_q,1, n_q,2, t_q,1, \ldots, t_q,n-2$ respectively, in the sense of the kinetic metric. Recall that the percussion vector $P_q$ verifies $P_q = p_{q,1}\nabla f_1(q) + p_{q,2}\nabla f_2(q)$ for some $p_{q,1}, p_{q,2} \in \mathbb{R}^+$. Then it is not difficult to show that the dynamical equation at the impact time $t_k$ is given by:

$$\begin{align*}
\sigma_{\dot{q}_{\text{norm}},1}(t_k) &= \eta_{q,1}^T P_q \\
\sigma_{\dot{q}_{\text{norm}},2}(t_k) &= \eta_{q,2}^T P_q \\
\sigma_{\dot{q}_{\text{tang}}}(t_k) &= t_q^T P_q = 0
\end{align*}$$

(8.10)

which is another way to write (8.7) (If $f(q) \in \mathbb{R}$, then there is only one component to $\dot{q}_{\text{norm}}$). Now note that if (8.4) is verified between 1 and 2, then (8.10) becomes

$$\begin{align*}
\sigma_{\dot{q}_{\text{norm}},1}(t_k) &= n_{q,1}^T p_{q,1} \\
\sigma_{\dot{q}_{\text{norm}},2}(t_k) &= n_{q,2}^T p_{q,2} \\
\sigma_{\dot{q}_{\text{tang}}}(t_k) &= 0
\end{align*}$$

(8.11)

$^5$Notice that since $M(q)\dot{q}$ is the generalized momentum of the system, the transformation can also be considered as a momentum transformation. However for control purposes, we shall not be interested in tracking a momentum desired value, but velocity and position desired values. Hence we prefer to consider the transformation in (8.9) as a velocity transformation. Note however that there might exist a feedback controller such that both the position $q$ and the momentum $M(q)\dot{q}$ track some desired trajectories. But manipulator feedback control has been not been attacked this way in the control and robotics literature. Since our switching controllers are based on these studies, we restrict ourselves to velocity and position tracking problems. The present study perhaps tells us that something else should be done for the control of complete robotic tasks.
i.e. the percussion component $p_{q,1}$ has no influence on the velocity $\dot{q}_{\text{norm},2}$ jump, and vice-versa. It is then possible and coherent to define a restitution rule (which is a generalization of that in (8.8)) as

$$
\begin{align*}
\dot{q}_{\text{norm},1}(t_k^+) &= -e_1\dot{q}_{\text{norm},1}(t_k^-) \\
\dot{q}_{\text{norm},2}(t_k^-) &= -e_2\dot{q}_{\text{norm},2}(t_k^-)
\end{align*}
$$

(8.12)

Note that from the definition of these quantities this can be rewritten as

$$
\dot{q}(t_k^+)^T \nabla_q f_1(q) = -e_1 \dot{q}(t_k^-)^T \nabla_q f_1(q)
$$

(8.13)

$$
\dot{q}(t_k^+)^T \nabla_q f_2(q) = -e_2 \dot{q}(t_k^-)^T \nabla_q f_2(q)
$$

(8.14)

We are now able to propose a general model for the system in (8.1) (8.2) (8.3).

### 8.3.2 A general form of the dynamical system

From the above developments, it is reasonable and natural to split the dynamical equations into three parts as follows (this is written for a codimension 1 constraint and can be easily generalized to the case $m \geq 2$ provided the above conditions are satisfied):

- **free-motion phases, motion control**

  $$
  M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u \\
  f(q) \geq 0
  $$

  (8.15)

  (8.16)

- **constrained-motion phases, force/position control**

  $$
  M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u + \lambda \frac{\partial f}{\partial q}(q) \\
  f(q) = 0 \text{ and } \lambda \geq 0
  $$

  (8.17)

- **transition phases**

  $$
  M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u \\
  f(q) > 0
  $$

  (8.18)

  $$
  M(q(t_k))\sigma_q(t_k) = P_q(t_k) \\
  f(q(t_k)) = 0
  $$

  (8.19)

  $$
  \dot{q}(t_k^+)^T \nabla_q f(q(t_k)) = -e\dot{q}(t_k^-)^T \nabla_q f(q(t_k))
  $$

  (8.20)

The set of equations in (8.15) through (8.20) must be completed in order to properly define the different phases: indeed in the free-motion phases, we allow for some collisions to occur, which explains the $\geq$ instead of $>$ in (8.16). These phases are
however distinct from the transition phases in the sense that they are defined such
that the fixed point of the closed-loop dynamical equations belongs to $\text{Int}(\Phi)$. In
other words, the fixed-point of the free-motion equation (8.15) (that will be given by
$\dot{q} \equiv 0$ i.e. $q^*(t) \equiv q_a$) is such that $f(q^*(t)) > 0$. On the contrary, it is assumed that
the closed-loop fixed point of (8.18) is such that $f(q^*(t)) < 0$. In fact as we shall
see later, the goal of the controller during the transition phases is to guarantee that
the dynamics in (8.18) through (8.20) define a discrete-time operator\(^6\) such that
$f(q(t_k))$ and $\dot{q}(t_k^+)T\nabla_q f(q(t_k))$ attain zero in finite time. We could have therefore
formulated it from this point of view. Note that (8.18) through (8.20) are
the closed-loop equivalent of (7.42).

Note that once the transition phase is finished (i.e. $\dot{q}(t_k^+)T\nabla_q f(q(t_k)) = 0$), two
situations can occur: either the applied forces are such that the system remains in
contact with $\partial \Phi$, or it detaches. This depends on the Lagrange multiplier $\lambda$ sign:
if $\lambda \geq 0$, constrained motion is insured. If $\lambda < 0$, the system detaches from the
constraint surface. In a one-degree-of-freedom case, this can be written as [232]:
$\dot{q} = \max[0, u]$ if $\dot{q}(t_k^+) = 0, q(t_k) = 0$ since the multiplier $\lambda = -u$. In our case such
outcomes are a consequence of the controller $u$. Since we do not a priori suppose
that a unique controller $u_i$ is applied, the complete dynamical system is given by
the above three phases, plus a control strategy of the following form:

Control strategy

\begin{align}
  u_i & \in \mathcal{U} \quad (8.21) \\
  u & = u_i \quad \text{if condition } C_i \text{ is true} \quad (8.22)
\end{align}

$\mathcal{U}$ is a set of controllers which stabilize the system when it evolves in one of the three
phases mentioned above\(^7\). Hence $\mathcal{U}$ contains position as well as force/positions
controllers. The conditions $C_i$ may be seen as a high-level controller, with possibly
an associated automaton [63]. They may be event-based or open-loop, see remark
8.19. As we shall see later, it is also possible to interpret $u$ as a unique controller,
discontinuous in $t$ and/or in the system's state (depending on the nature of the
conditions $C_i$).

Remark 8.5 It is clear that the aim of the robotic task is to make the manipu-
lator track some desired trajectories. During free and constrained-motion phases
one classically defines desired position, velocity and interaction force trajectories.
The controller $u$ has to be designed such that the tracking errors vanish asymp-
totically. The system's behavior during the transition phase will in general be
quite different. Indeed in some tasks it may be required that impact times verify
$t_{k+1} = t_k + T$ for some $T > 0$ with bounded trajectories on $(t_k, t_{k+1})$. (one can
think of a "hammer"-like task). This also implies that $u$ in (8.18) is designed such
that the sequence of impact times $\{t_k\}$ does exist. Note that in general the con-
troller stabilizing free-motion phases will not be suitable for the transition phases

\(^6\)The "sampling-times" being of course defined from the impact times.

\(^7\)One of the main conclusions of the experimental work in [560] is that three distinct controllers
have to be used for the control of a complete task.
CHAPTER 8. FEEDBACK CONTROL

objectives. In this section, we will consider only robotic tasks which involve alternatively free and constrained-motion phases. As we pointed out in the introduction of this part, although the model in (8.16) through (8.22) might represent other classes of mechanical systems, we do not consider such extensions. Hence the goal of the controller during the transition phases will be to guarantee that the manipulator stabilizes on the constraint surface \( f(q) = 0 \) in finite time. In other words we shall require that the sequence \( \{t_k\} \) has a finite accumulation point \( t_\infty < +\infty \) with \( f(q(t_\infty)) = 0 \). Except if \( e = 0 \), \( \{t_k\} \) is an infinite sequence. \( t_\infty < +\infty \) implies that \( e < 1 \), i.e. there is a dissipation of energy at collisions.

Remark 8.6 Using the generalized coordinates transformation described in [324], see remark 6.11 in chapter 6, the unilateral constraint \( f(q) \geq 0, \ f(q) \in \mathbb{R}^m \), can be rewritten in a new set of coordinates \( X \triangleq \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \), where \( x_1 \in \mathbb{R}^m \) and the constraint \( f(q) = 0 \) becomes simply \( x_1 = 0 \). This now well-known transformation relies on a suitable partition of the coordinates \( q \) and may be assumed for convenience to be global. The constraints \( f_i(q) \) are also assumed to be independent. Therefore the constraint \( f(q) \geq 0 \) can be rewritten as \( x_1 \geq 0 \), so that the Euclidean normal vector to the constraint hypersurface \( x_{1,i} = 0 \) is simply the \( i \)-th unit vector \( e_i \in \mathbb{R}^n \).

It is noteworthy that although the rest of the coordinates \( x_2 \in \mathbb{R}^{n-m} \) represents the motion of the system along the tangential part of the constraint, \( \dot{x}_2 \) does not in general remain continuous at the impact times if one adopts the generalized Newton's restitution rule described in (8.8)-(8.14): indeed one has to compute the jump in \( \dot{x}_1 \) from the restitution rule, and then compute the jump in \( \dot{x}_2 \) from (8.7), i.e. the algebraic dynamical equation at the shock instant, see remark 6.11. We shall come back on these calculations in section 8.6.2. In fact the velocity transformation in (8.9) can be performed starting from any set of generalized coordinates, in particular \( X \). However one concludes that for simplicity (but not without loss of generality since the transformation is assumed to hold globally) the unilateral constraints could be written as \( q_i \geq 0, \ 1 \leq i \leq m \). Also the restitution rule in (8.19) becomes \( \dot{q}_i(t_k) = -e_i \dot{q}_i(t_k) \) when \( m \geq 1 \). Notice that given this assumption, the condition of orthogonality in (8.4) becomes \( e_i^T M^{-1}(q) e_j = 0, \ i, j \in \mathbb{I}, i \neq j \), i.e. the \( ij \)-th entry of \( M^{-1}(q) \) should equal zero [232]. Let us recall that the developments in (8.9) through (8.14) prove that such restitution rule allows to compute the \( n + m \) unknowns of the problem ( \( n \) postimpact velocities \( \dot{q}(t_k^+) \) and \( m \) percussion components \( p_{q,i} \) ) from preimpact velocities and restitution coefficients. Moreover if the basis \( (n_q, t_q) \) is orthonormal, then the kinetic energy loss at impacts is given by \( T_L = T(t_k^+) - T(t_k^-) = \sum_{i=1}^{m} \frac{1}{2} (e_i^2 - 1) (q_{norm,i}(t_k^-))^2 \), see subsection 6.2.1. In particular \( T_L \leq 0 \) for any preimpact velocity implies \( e_i \in [0,1], \ 1 \leq i \leq m \).

8.3.3 The closed-loop formulation of the dynamics

The equation in (8.18) represents the smooth dynamics between impacts, i.e. on intervals \( (t_k, t_{k+1}) \). In general the closed-loop state variables contain the position
tracking error $\ddot{q} = q(t) - q_d(t)$, where $q_d(t) \in C^2[\mathbb{R}^+]$ represents some desired motion. Then the unilateral constraint $f(q) \geq 0$ can be written as $f_t(\ddot{q}) \triangleq f(\ddot{q} + q_d) \geq 0$. In other words, the invariant constraint can be written in closed-loop form as a time-varying constraint. The restitution rule in (8.8) can then be rewritten as

$$\nabla f^T \ddot{q}(t_k) = -e\nabla f^T \ddot{q}(t_k^-) - (1 + e)\nabla f_t$$

(8.23)

As we saw in remark 6.9, since the constraint is time-varying, the restitution rule must incorporate its velocity at the impact time. This is represented in the last term of the right-hand-side of (8.23). From the definition of the time-varying closed-loop unilateral constraint $f_t(q)$, it is easy to verify that $\nabla f_t = \nabla f$ and that $\nabla f^T \dot{q}_d = \nabla f_t$, so that (8.8) and (8.23) are equivalent. Let us prove this statements. First let us rewrite (8.23) as

$$\nabla f^T \ddot{q}(t_k^+) = -e\nabla f^T \ddot{q}(t_k^-) + (1 + e) \left[ \nabla f^T \dot{q}_d(t_k) - \nabla f_t(t_k) \right]$$

(8.24)

Now $f_t(\ddot{q}) = f(q)$ by definition. For any $q(t)$ we have $\ddot{q}(t) + q_d(t) = q(t)$. Let us compute $\frac{\partial f_t}{\partial q}$ at some $\ddot{q}(0)$. Note that $\ddot{q} + q_d = \ddot{q}_0$ for the corresponding $q_0$ since there is a bijective relation between $\ddot{q}_0$ and $q_0$ at time $t$. Hence we get

$$\frac{\partial f_t}{\partial q}(\ddot{q}_0) = \lim_{q \to \ddot{q}_0} \frac{f(q) - f(q_0)}{q - \ddot{q}_0} = \lim_{q \to \ddot{q}_0} \frac{f(q + q_d) - f(q_0 + q_d)}{q - \ddot{q}_0}$$

(8.25)

$$= \lim_{q \to q_0} \frac{f(q) - f(q_0)}{q - q_0} = \frac{\partial f}{\partial q}(q_0)$$

Therefore $\nabla f_t = \nabla f$. Let us prove the second assertion:

$$\frac{\partial f_t}{\partial q}(t_0) = \lim_{t \to t_0} \frac{f(q(t) + q_d(t)) - f(q(t_0) + q_d(t_0))}{t - t_0} = \lim_{t \to t_0} \frac{f(q(t)) - f(q(t_0))}{t - t_0}$$

(8.26)

$$= \frac{\partial f^T q_d}{\partial q} = \nabla f^T \dot{q}_d = \nabla f_t \dot{q}_d$$

where we used the fact that $v(t) = q(t)$. Hence $\nabla f^T \dot{q}_d(t_k) - \nabla f_t(t_k) = 0$.

From a more general point of view, let us denote the dimension of the closed-loop vector $x(t)$ as $l \geq 2n$. Notice that we could associate a closed-loop vector $x_k$ to each one of the different phases that constitute the hybrid dynamical system. This path will be pursued in section 8.7. However as we shall see in section 8.4, it is also possible to consider that the total closed-loop state vector is given by only one variable $x(t)$, including the fact that during some phases, certain components may be frozen. This is the case for instance when the system is in permanent contact with the constraint: one can assume that the state of the system is reduced, which is a classical way of proceeding in mechanics of holonomically constrained systems. One can also just consider that some parts of the state are zero, for instance $x_1 = \dot{x}_1 = 0$ if the MacClamroch-Wang’s coordinates [324] are used, and study the evolution of $x(t)$ taking into account this fact. This is contained in (8.17). Then we have $x \in \Phi_t \times \mathbb{R}^n \times \mathbb{R}^{l-2n}$. $\Phi_t \subset \mathbb{R}^n$ denotes the domain within which $\ddot{q}(t)$ evolves. It is obviously time-varying if $q_d$ is a time-function. At $t_k, x(t_k^-) \in \partial \Phi_t \times -V_t(q) \times \mathbb{R}^{l-2n}$,
whereas $x(t^+_{k}) \in \partial \Phi_t \times V_t(q) \times \mathbb{R}^{l-2n}$. $\Phi_t$ is defined by the inequalities $f(q + q_d(t)) \geq 0$. In order to clarify the meaning of the domain $V_t(q)$, let us define the relative velocity 
\[
\dot{q}_r = \left( \begin{array}{c} \nabla_q f_t^T \dot{q} - \nabla f_t \\ t_q^T \dot{q} \end{array} \right),
\]
where $t_{q}^{m+1} \leq i \leq n$, $f(q) \in \mathbb{R}^m$. Then
\[
V_t(q) = \{ v \in \mathbb{R}^n : v^T \nabla_q f(q) - \nabla f_t \leq 0 \}.
\]
It is clear then that $\dot{q}_r(t^-_{k}) \in -V_t(q(t_k))$ whereas $\dot{q}_r(t^+_{k}) \in V_t(q(t_k))$.

In conclusion it is possible to transport all the open-loop formulation of the shock dynamics (that we described thoroughly in chapters 5 and 6) in a closed-loop formulation, with time-varying unilateral constraints and appropriate dynamic restitution rules. Although the usefulness of such an operation is not clear, it is nevertheless noteworthy that time-varying constraints naturally appear in controlled mechanical systems. This is due to the fact that we need to transform the open-loop state in a closed-loop error state when dealing with control and stability.

8.3.4 Definition of the solutions

In chapters 2 and 5 we have seen that the solution $q(t)$ of an impact problem possesses in general derivatives which are not piecewise continuous but rather of local bounded variation in time and right-continuous (in short $RCLBV$). This also allows to define the acceleration as a bounded positive measure since it is the derivative of a $RCLBV$ function [562]. Nevertheless it clearly appeared (at least we hope it did!) that in the general case of a system as in (8.1) (8.2) (8.3), such assertions are not trivial and must be proved. The most advanced results in this direction can be found in [416], see section 2.2. In particular, let us recall that theorem 2.1 in chapter 2 requires that a) the system evolves in a convex domain of the state space, i.e. the region $\Phi$ defined by inequalities (8.2) be convex, b) with a regular (i.e. twice differentiable) boundary, and c) that the external action $u(t)$ on the system be continuous in time. This may not be verified for systems like in (8.1) (8.2) (8.3). However we have seen in remark 8.6 that the coordinate change like the one proposed in [324] and suitable feedback control law allow to put the closed-loop system in the framework developed in [416]. In particular this allows to get a convex domain $\Phi$. Although some dynamical problems with unilateral constraints may not possess any solution in such spaces, we conjecture that under the restrictions on the unilateral constraints we impose (i.e. either uniqueness or multiplicity but for orthogonal constraints), existence is assured. The convexity of $\Phi$ is however not a real obstacle as long as locally, existence and uniqueness of a projection on $\partial \Phi$ is assured. Multiplicity of the constraints (i.e. regularity of $\partial \Phi$) is a more serious problem: existence has been proved only for zero energy loss at impacts [417]; this is not sufficient here because we need finite time stabilization results, which cannot be obtained when $\varepsilon = 1$. Continuity of the external action in $t$ may also be relaxed to measurability in the Lebesgue sense [418], and with Carathéodory-like conditions, since the crucial point is to guarantee existence of solutions of a penalizing problem.
8.4. STABILITY ANALYSIS FRAMEWORK

(see [416] theorem 2). Hence in the following, we shall always assume that \( \dot{q} \in RCLBV \). Furthermore notice that this has nice consequences for stability analysis purposes, since RCLBV functions possess a countable set of discontinuity points on any compact time interval, see appendix C. It is consequently natural to associate a discrete time system (or impact Poincaré map) to such impacting systems, see chapter 7, section 7.1.4. This is a natural extension in the set of solutions of bounded variation of the results in [27] (see chapter 7, section 7.1) who assume only piecewise continuous solutions.

Another problem appears at the switching times between the different controllers \( u_i \), see (8.21). Two situations may be considered: either the switching period has a strictly positive measure, or it has a zero measure. It seems realistic to assume zero measure, since in practice the switching will generally be realized via software, and consequently will be almost instantaneous. Hence what happens “during” the switch will be disregarded in the stability analysis. Finally which sense should we give to the control law at the switching times? Mills and Lokhorst [360] adopt the point of view of differential inclusions, i.e. assume that \( u \in \left[ u_{\text{min}}, u_{\text{max}} \right] \) at the switching times. Then the overall system solutions have to be considered as reachable sets, no longer time functions, and existence as well as uniqueness, stability results have to be generalized, see [414] [154]. As noted in [360], no general mathematical tool seems available in the mathematical literature to study differential inclusions with measures in the right-hand-side, or with state jump conditions (\(^8\)). It would be possible to define control strategies such that no collision occurs at the switches. We however prefer to assume in this study that at a switching time \( T_s \) the controller takes a certain (unknown) value between \( u(T_s^-) \) and \( u(T_s^+) \). Actually we formulate the problem with a time-discontinuous controller, supposed for instance to be right-continuous. This is much less elegant from a theoretical point of view, but allows to get rid of a problem with few practical consequences on the stability analysis, as long as the switching period is supposed to be zero.

8This is an open problem. We already discussed about this fact in section 1.4.1 in relationship with a coordinate change in MDE’s.

8.4 Stability analysis framework

In chapter 7, we have made a thorough review of the different manners of proving stability results for impacting systems. We concluded in particular that to consider the stability of a complete robotic task with rigid obstacles, we should have to reconsider and generalize the existing analytical tools, see sections 7.1, 7.1.1, 7.1.2, 7.1.4. This is the point we reach now. The system in (8.15)-(8.21) is a complex hybrid dynamical system [63], which involves continuous as well as discrete time phases. Stability criteria have been proposed for simple hybrid systems, see [63]. However they do not apply to more complicated system as in (8.15)-(8.21). We choose a time-domain approach following the spirit of Lyapunov’s second method. The stability analysis will therefore be based on the choice of a suitable unique pos-
itive definite function $V$ of the system's state. Although this seems the simplest and the most natural way to proceed, it is in fact not clear which conditions of variation of $V$ should be required. In general the controllers may be dynamic output or state feedback laws. Hence the equations in (8.15)-(8.21) do not represent the whole closed-loop system. Note that for the moment the representation of the transition phase in (8.18) and (8.19) is not very tractable. Indeed it is amenable for for stability analysis via the tools developed in [27] which are well-suited when one wants to stabilize the manipulator close to the obstacle, see example 7.1. In this case accidental collisions may occur between the robot's tip and the constraint surface. Such stability criteria guarantee that these collisions do not destroy Lyapunov stability of the closed-loop system. For a system submitted to unilateral constraints, this corresponds to $x^* \in \text{Int}(D)$, where $D = \Phi \times \mathbb{R}^n \times \mathbb{R}^{l-2n} \ni (q, \dot{q}, z)$, where $z$ comes from eventual dynamic state feedback, and $x^*$ denotes the closed-loop error system fixed point (that may be a time-varying trajectory). Notice that due to the form of the unilateral constraints this may correspond to $f(t, x^*) \neq 0$ and $I_k(x^*) = 0$ in (7.1), i.e. $x^*$ is given by the discrete dynamics and the system is defined as in 7.1. This again is the case for the bouncing ball where the constraint is $x \geq 0$, $x = \left( \begin{array}{c} q \\ \dot{q} \end{array} \right)$, $f(t, x) = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) x + \left( \begin{array}{c} 0 \\ -g \\ -q \end{array} \right)$ and $I(x(t_k^*)) = \left( \begin{array}{c} 0 \\ -(1+e)q(t_k^*) \end{array} \right)$.

The fixed point is nevertheless given by $q = \dot{q} = 0$ since the dynamics of the bouncing ball must be completed by $\ddot{q} = \max[0, f(t, 0)]$ when $q(t_k^*) = 0$, or more exactly $\ddot{q}(t_k^*) = 0$ (for instance after the first shock if $e = 0$). This allows to retrieve the algebraic dynamical equation of the constrained case. As noted in section 7.1, stability of such systems with impulse effects is not treated in [27]. As we noted in chapter 7, classical stability concepts on metric spaces [619] do no carry out to this situation where it is not possible to define neighborhoods (open domains) of $x^*$ on $\partial \Phi \times \mathbb{R}^n \times \mathbb{R}^{l-2n}$. Moreover the fact that the fixed point is not given by the continuous vector field but rather by the algebraic impact equations, renders the results in [27] unapplicable, see remark 7.7. A way to overcome this problem for the analysis of the transition phase is to study the impact Poincaré map\(^{10}\) associated to (8.18) and (8.19). In general such maps are difficult to be obtained explicitly. However one may use the control input $u$ in (8.18) to simplify the smooth dynamics (for instance feedback linearization) and make it possible to get explicitly the discrete time operator associated to (8.18) (8.19).

**Remark 8.7** Another way to avoid this problem is to use the Zhuravlev-Ivanov nonsmooth changes of coordinates which transform systems with unilateral constraints as in (8.1)-(8.3) into systems with time-continuous trajectories, see subsection 1.4.2. Let us reconsider example 1.7. Assume we are given the same second-order linear system, but with a controller:

\[ u(t, q, \dot{q}) = \ddot{q} - \lambda_1 \dot{q} - \lambda_2 \dot{q} \]

\(^9\)See remark 7.4 concerning this topic.

\(^{10}\)With some abuse of name since Poincaré maps are strictly speaking defined from periodic trajectories [1], whereas here it will concern solutions converging to zero in finite time (but with an infinity of rebounds).
Hence the closed-loop equation is simply $\ddot{q} + \lambda_1 \dot{q} + \lambda_2 \dot{q} = 0$, with $q \geq 0$. This can be rewritten with the nonsmooth coordinate change as in example 1.7:

$$\dot{s} + \lambda_1 s + \lambda_2 s = f(t) \text{sgn}(s)$$

(8.28)

where $f(t) = \dot{q}d + \lambda_1 \dot{q}d + \lambda_2 \dot{q}d$. One notices that the fixed point that is now studied in (8.28) is $s = \dot{s} = 0$ which indeed lies on the domain boundary. Depending on $q_d(t)$, it may be possible to take advantage of the Zhuravlev-Ivanov nonsmooth transformation to study the closed-loop dynamics.

In summary, systems as in (7.1) and mechanical systems with unilateral constraints on the position, essentially differ by the fact that the state is not assumed to evolve in a domain with a boundary in the former (hence extension of Lyapunov stability concepts [27] [316] [452] [327] for measure differential equations), whereas it is for the latter.

To clarify the proposed stability criterion, and to overcome those difficulties, we introduce the following definitions: let us split the time axis into intervals $\Omega_k$ and $I_k$, corresponding to smooth phases (during which collisions may nevertheless occur at times $t = t_k$ (11) for motion phases in (8.15), but the closed-loop equations fixed-point belongs to $\text{Int}(\Phi)$) and transition phases respectively (the goal is to obtain stabilization of the system on the surface $\partial\Phi$). From the hybrid dynamical systems point of view, the $\Omega_k$'s correspond to the continuous time phases and the $I_k$'s to discrete-time phases. As we shall see, $I_k$ does not necessarily correspond to the rebounds and stabilization on the constraint phase, but more generally contains it. Even in the case of plastic impact (e = 0) in general $\lambda[I_k] > 0$ (in Lebesgue measure). Note that $\mathbb{R}^+ = \bigcup_{k \in K} \Omega_k \cup \bigcup_{k \in K_d} I_k$. $K_c$ and $K_d$ are the sets of indices for phases $\Omega_k$ and $I_k$ respectively. They may be finite or infinite. We denote $\Omega_k = [T^k_0, T^k_f]$ and $I_k = [t^k_l, t^k_f]$. Note that $\Omega_k$ corresponds to free as well as constrained motion phases. Hence there cannot be more than two sequential intervals $\Omega_k, \Omega_{k+1}$, since transition from free to constrained motion phases must be an impact phase. Hence we have for a typical task:

$$\mathbb{R}^+ = \Omega_0 \cup I_0 \cup \Omega_1 \cup \Omega_2 \cup I_1 \cup \ldots \Omega_{2k-1} \cup \Omega_{2k} \cup I_k \ldots$$

(8.29)

We now introduce the following lemmas (12). Here $x(t) \in \mathbb{R}^l$, $l \geq 2n$, denotes the closed-loop state vector and may be different from $(q, \dot{q})$, mainly because of dynamic state feedback controllers, and tracking purposes. In general $x(t)$ has the following form: $x^T = (q^T, \dot{q}^T, z^T)$ (13).

11The impact times are generically denoted as $t_k$, similarly as in the foregoing chapters. This does not mean that a time $t_k$ is related to the domain $\Omega_k$. The subscript $k$ is a dummy variable associated to the different quantities that we need to define, i.e. the different time intervals, their upper and lower bounds, the impact times, etc. Moreover to simplify the notations, we employ $t_k$ for all the impact phases.

12See remark 8.12.

13See also remark 8.18 concerning the definition of the closed-loop tracking error $\dot{q}$. 

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Hence the closed-loop equation is simply $\ddot{q} + \lambda_1 \dot{q} + \lambda_2 \dot{q} = 0$, with $q \geq 0$. This can be rewritten with the nonsmooth coordinate change as in example 1.7:
Lemma 8.1 (weak stability)

Let $x(t)$ denote the state of the closed-loop system (8.15)-(8.21), with $x(t) \in R^{\text{CLBV}}$. Let $V(x)$ be a positive definite function, continuous in $x$, satisfying $\alpha(||x||) \leq V(x) \leq \beta(||x||)$ where $\alpha(.)$ and $\beta(.)$ are class K functions\(^{14}\). Assume that on intervals $I_k$, $V(t_k^0) \leq V(t_k^{k-1})$ along the system’s trajectories, $V$ is uniformly bounded on $I_k$, and $\lambda[I_k] < +\infty$ for all $k \geq 0$. Define $\Omega$ as the complement of $I = \cup_k I_k$, so that $\lambda[\Omega] = +\infty$

a) If on $\Omega$, $V(x(t)) \leq -\gamma(||x(t)||)$ for $t \neq t_k$ along system’s trajectories for some class K function $\gamma(.)$, and $\sigma_V(t_k) \leq 0$, then $x(t) \to 0$ as $t \to +\infty$, $t \in \Omega$.

b) If $x^T = [x_1^T \ x_2^T]$, $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$, $V(x(t)) \leq -\gamma_1(||x_1(t)||)$ for $t \neq t_k$, on $\Omega_1$, $V(x(t)) \leq -\gamma_2(||x_2(t)||)$ for $t \neq t_k$ on $\Omega_2$, $\sigma_V(t_k) \leq 0$ for $t_k \in \Omega$, $||x_1||$ nonincreasing on $\Omega_2$, $||x_2||$ nonincreasing on $\Omega_1$, $\lambda[\Omega_1] = +\infty$, $\lambda[\Omega_2] = +\infty$ and $\Omega = \Omega_1 \cup \Omega_2$, then $x(t) \to 0$ as $t \to +\infty$, $t \in \Omega$.

c) If $\lambda[\Omega_1] < +\infty$, $\lambda[\Omega_2] = +\infty$, $\lambda[I] < +\infty$, $\Omega = \Omega_1 \cup \Omega_2$, $x_1(t)$ is constant on $\Omega_2$, $V(x_2(t)) - V(x_2,0) \geq 0$ for any $x_2 \in R^{n_2}$, and $\hat{V}(x(t)) \leq -\gamma_2(||x_2(t)||)$ for $t \neq t_k$, $\sigma_V(t_k) \leq 0$, then $x_2(t) \to 0$ as $t \to +\infty$.

* proof of lemma 8.1

a) Assume that $\lim_{t \to +\infty, t \in \Omega} V(x(t)) = \delta > 0$. Since $\beta(||x||) \geq V(x)$ for all $t \geq 0$, and $V$ is nonincreasing on $\Omega$, then $\delta = \min_{t \in \Omega} V(t)$, and $||x(t)|| \geq \beta^{-1}(\delta) > 0$ for all $t \in \Omega$. Since $t \in \Omega$ we can write for a certain $n \in N$(assuming the initial time is in $\Omega)$:

$$V(t) - V(0) = V(t) - V(t_k^0) + V(t_k^0) - V(t_k^{k-1}) - V(t_k^{k-1}) + V(t_k^{k-2}) + \ldots + V(t_k^0) - V(0)$$

Consequently, since $V(t_k^j) \leq V(t_k^0)$ we obtain

$$V(t) - V(0) \leq - \sum_{j=1}^{n} \int_{t_j}^{t_{j+1}} \gamma(||x(\tau)||) d\tau - \int_0^t \gamma(||x(\tau)||) d\tau - \int_0^0 \gamma(||x(\tau)||) d\tau + \sum_k \sigma_V(t_k) \leq -\gamma_0 \beta^{-1}(\delta) \lambda[\Omega_{[0,t]}]$$

Now we can always find $t$ (or $n$) such that for any $V(0) < +\infty$ and for any $\delta > 0$ the inequality $V(t) + \gamma_0 \beta^{-1}(\delta) \lambda[\Omega_{[0,t]}] \leq V(0)$ is not verified. Hence by contradiction we deduce that $\delta = 0$, and since $||x(t)|| \leq \alpha^{-1} o V(x(t))$ it follows

\(^{14}\)i.e. $\alpha(0) = \beta(0) = 0$, they are strictly increasing and radially unbounded on $[0, +\infty)$. 
that \( \lim_{t \to \infty, t \in \Omega} x(t) = 0 \). Note that if \( t \notin \Omega \) the reasoning fails because for \( t \in I \) the terms \( V(t) - V(t_k) \) may be positive.

The case b) can be similarly proved, by noting that \( \| x(t) \| \geq \beta^{-1}(\delta) > 0 \) for all \( t \in \Omega \) implies that necessarily \( \| x_1 \| \geq \delta_1 > 0 \) and/or \( \| x_2 \| \geq \delta_2 > 0 \) for some \( \delta_1 \) and \( \delta_2 \). Then using that \( \| x_1 \| \) and \( \| x_2 \| \) do not increase on \( \Omega^2 \) and \( \Omega^1 \) respectively, and that these time intervals have infinite Lebesgue measure, one deduces that if either \( \delta_1 \) or \( \delta_2 \) is strictly positive (i.e. \( \delta > 0 \)), then a contradiction appears and necessarily \( \delta = 0 \).

c) From the assumptions there exists a time \( T < +\infty \) such that for all \( t \geq T \), \( x_1(t) \) is a constant vector \( X_1 \). Let us define the function \( V_{X_1}(x_2) = V(X_1, x_2) - V(X_1, 0) \). Then for any \( \delta \geq 0 \) the sets \( V_{X_1}^{-1}(\delta) = \{ x_2 : V_{X_1}(x_2) \leq \delta \} \) are equal to the sets \( V^{-1}(\delta + V(X_1, 0)) \cap \{ x_1 = X_1 \} \subset \mathbb{R}^n \). Hence they are bounded and closed, thus compact. Now \( V_{X_1}(0) = 0, \ V_{X_1}(x_2) \geq 0, \ V_{X_1}(t) = V(t) \leq -\gamma_2(\| x_2(t) \|) \) for \( t \neq t_k \), and \( \sigma_{V_{X_1}}(t_k) = \sigma_V \leq 0 \). The conclusion follows similarly as in a). The conditions apply when for instance \( V(x_1, x_2) = V_1(x_1) + V_2(x_2), \ V_1(x_1) \geq 0, \ V_2(x_2) \geq 0 \).

\[ \nabla \nabla \nabla \]

Note that since \( V(x) \) is continuous in \( x \), then \( V(t) \in RCLBV \). It is thus justified to use the jumps of \( V \) at the impacts times in the stability analysis. Recall also that at such times \( t_k \), the time derivative of \( V(t) \) is calculated via a generalized chain rule [562], i.e. \( \dot{V}(t_k) = \sigma_V(t_k) \delta_{t_k} \). To clarify again the stability concept, let us rewrite the closed-loop dynamics as

\[
\begin{cases}
    t \in \Omega_{2k} : & \dot{x} = G_{nc}(x,t) \\
    t \in \Omega_{2k+1} : & \dot{x} = G_c(x,t) \\
    t \in I_k : & \begin{cases}
                \dot{x} = G_l(x,t) \\
                \sigma_x(t_k) = G_k(x(t_k^-), t_k) \\
                f(q(t_k)) = 0
            \end{cases}
\end{cases}
\] (8.32)

We did not indicate the possible collisions in \( \Omega_{2k} \) for simplicity. Notice that it is implicitly understood in lemma 8.1 conditions that the vector fields \( G_{nc} \) and \( G_c \) possess the same unique fixed point. Let \( x^* = 0 \) be this unique point satisfying \( G_{nc}(x^*, t) = G_c(x^*, t) = 0 \). Hence weak stability implies that if \( x(\tilde{t}) = 0 \) for some \( \tilde{t} \in \Omega \), then for all \( t \geq \tilde{t}, t \in \Omega, x(t) = 0 \). In a sense, \( x^* \) is required to be a fixed point of the closed-loop system only on \( \Omega_k \).

Remark 8.8 The term weak stability is also used in the context of differential inclusions [154], where it means that there is at least one function in the set of solutions (one selection) that is stable. This is clearly quite different from what we deal with in lemma 8.1.

In the next lemma we introduce stronger conditions for the behaviour of the system at the impact times \( t_k \).
Lemma 8.2 (strong stability)

Let $x(t)$ and $V(x)$ be as in lemma 1, and satisfy the conditions of lemma 1 a), b) or c). Assume furthermore that on $I_k$,

i) $\sigma V(t_k) \leq 0$,

ii) $V(t_{k+1}) \leq V(t_k)$,

iii) $V$ is uniformly bounded and time continuous on $I_k - \bigcup_k \{t_k\}$ where the sequence $\{t_k\}$ exists and has a finite accumulation point. Then the results of lemma 1 are true.

Assume further that on $I_k$, $q_d \equiv 0$, and that the transformation $G_i : x \mapsto \bar{x}_i^T = (f_1(q), q_2, \ldots, q_n, q^T \frac{\partial f_k}{\partial q}, q_2, \ldots, q_n, z)^T$ preserves the properties of the function $V$ (i.e. $\bar{V}(\bar{x}_i) \equiv V(x)^{-1}(\bar{x}_i)$ is defined globally and satisfies the above requirements i, ii, iii). Then the closed-loop impact Poincaré map (15) $P_{\Sigma,i} : \bar{x}_{\Sigma,i}(t+1) \mapsto \bar{x}_{\Sigma,i}(t_{k+1}^+)$, $P_{\Sigma,i}(0) = 0$, where $\bar{x}_{\Sigma,i} \in \mathbb{R}^{d-1}$ and $\Sigma_i$ is the impact Poincaré section defined as $\Sigma_i = \{x : f_i(q) = 0\}$, is Lyapunov stable with the Lyapunov function $V_{\Sigma,i}$ (the restriction of $V$ to the section $\Sigma_i$).

For the definition of the impact Poincaré maps $P_{\Sigma,i} : \Sigma_i \rightarrow \Sigma_i$ see subsection 7.1.4 in chapter 7.

The proof of lemma 8.2 follows from lemma 8.1 and [164], see subsection 7.1.4, theorem 7.2. Indeed by assumption we get $V(t_{k+1}) \leq V(t_k)$. Hence since $V_{\Sigma,i} = \bar{V}(\bar{x}_i)$, we deduce that $V_{\Sigma,i}(t_{k+1}^+) \leq V_{\Sigma,i}(t_k^+)$ so that $\bar{x}_{\Sigma,i}^+ = 0$ is a globally Lyapunov stable fixed point of $P_{\Sigma,i}$. Although $V$ is designed as a continuous-time function, the above conditions assure that $V_{\Sigma,i}$ qualifies as a discrete-time Lyapunov function for the mapping $P_{\Sigma,i}$. For $P_{\Sigma,i}$, it is perfectly legitimate to speak of Lyapunov stability, contrarily to the section map $P : x(t_k^+) \mapsto x(t_{k+1}^+)$ (16). Notice that we have fixed $q_d \equiv 0$ in lemma 8.2, in order to be able to properly define the variable change $G_i$ which hence does not explicitly depend on time. More generally, if the functions $f_i(q)$, $1 \leq i \leq m$, involve only $q_1, \ldots, q_r$, $r \leq n$, then it is sufficient to have $q_d, i \equiv 0, 1 \leq i \leq r$. If one applies (as we shall do) a first variable change to get the constraints in the simple form $q_i \geq 0, 1 \leq i \leq m$, then clearly $G_i$ can be chosen as the identity mapping.

In summary, following the developments in chapter 7 on the definition and calculation of the impact Poincaré map $P_{\Sigma,i}$, we define the closed-loop state vector $x$, the transformed closed-loop state vector $x_i$, and the closed-loop Poincaré state vector

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15 The preimpact values can be chosen as well to define this mapping. Notice that the fact that $t_\infty < +\infty$ is not a consequence of the stability of $P_{\Sigma,i}$. It has to be proved.

16 As we already discussed, the section map $P$ is not really suitable for the study of Lyapunov stability, see remark 7.4. But it may be used for other purposes, noting that $P$ has as many Lyapunov exponents as the flow with collisions $\varphi^*_T$. For instance Wojtkowski [586] studies the instability of an $n$-balls system showing the existence of nonvanishing Lyapunov exponents of $P$. 
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\[ x = \begin{pmatrix} q - q_d \\ \dot{q} - \dot{q}_d \\ z \end{pmatrix} \in \mathbb{R}^l, \quad \bar{x}_i = \begin{pmatrix} f_i(q) \\ q_2 \\ \vdots \\ q_r \\ q_{r+1} - q_{d,r+1} \\ \vdots \\ q_n - q_{d,n} \\ \dot{q}_r \\ \dot{q}_{r+1} - \dot{q}_{d,r+1} \\ \vdots \\ \dot{q}_n - \dot{q}_{d,n} \\ z \end{pmatrix} \in \mathbb{R}^l, \quad \bar{x}_{E,i} = \begin{pmatrix} q_2 \\ \vdots \\ q_r \\ q_{r+1} - q_{d,r+1} \\ \vdots \\ q_n - q_{d,n} \\ \dot{q}_2 \\ \cdots \\ \dot{q}_r \\ \dot{q}_{r+1} - \dot{q}_{d,r+1} \\ \vdots \\ \dot{q}_n - \dot{q}_{d,n} \\ z \end{pmatrix} \in \mathbb{R}^l \]

(8.33)

The definition of \( \bar{x}_{E,i} \) makes sense only at impact times since it is attached to \( P_{E,i} \).

The fixed points of the closed-loop dynamical equations on \( \Omega \) and \( I \) are given by \( x = x^* = 0 \) and \( \bar{x}_{E,i} = \bar{x}^*_{E,i} = 0 \) respectively. Notice that it suffices that \( G_i \) be a global diffeomorphism for the results to hold. Indeed in this case by construction \( \dot{V}(\bar{x}_i) \) exists globally and \( \dot{V}(\bar{x}_i) = V(x) \) for all \( t \geq \tau_0 \).

**Remark 8.9** We could have stated lemma 2 differently. For instance, we could have stated conditions i and ii simply as i) \( V(t_{k+1}^+) \leq V(t_k^+) \). Then there is equivalence between conditions i and iii and the fact that the impact Poincaré maps \( P_{E,i} \) are Lyapunov stable with Lyapunov functions \( V_{E,i} \). This equivalence holds if the mappings \( G_i \) are (globally) diffeomorphic, which imposes some conditions on the functions \( f_i(q) \), see also remark 8.10. Also we have required only stability, not asymptotic stability of the maps \( P_{E,i} \). In fact the asymptotic property is contained in condition iii since we require that \( \{t_k\} \) possesses a finite accumulation point. As we shall see later, proving asymptotic stability of \( P_{E,i} \) may not be obvious even in simple cases.

**Remark 8.10** The way \( \bar{x}_i \) is defined presupposes that in order for \( G_i \) to be diffeomorphic, one may need a reordering of the variables. Clearly, if for instance \( f_2(q) = q_2 \), then it makes no sense to define \( \bar{x}_2 \) as above. But then trivially either one places \( f_i(q) \) at the \( i \)-th component of \( \bar{x}_i \), or reorders the generalized coordinates \( q \).
The closed-loop system can now be written as

\[
\begin{cases}
  t \in \Omega_{2k} : & \dot{x} = G_{nc}(x, t) \\
  t \in \Omega_{2k+1} : & \dot{x} = G_c(x, t) \\
  t \in I_k : & \left\{ \begin{array}{l}
  P : x(t_k^+) \mapsto x(t_{k+1}^+) \\
  \text{or} & P_{E,i} : \bar{x}_{E,i,k} \mapsto \bar{x}_{E,i,k+1}
  \end{array} \right.
\end{cases}
\]  

(8.34)

Recall that it is understood in the formulation of lemma 8.2 that the controller stabilizes the system in finite time on the constraint, see iii. This is in fact different from the Lyapunov stability of the mappings \( P_{E,i} \). The interest for the strong stability is that if i) through iii) are verified, then the whole stability proof for the closed-loop system in (8.34) is led with a unique \( V \), up to a possible transformation \( G(t) \): it is indeed generally admitted in control theory that it is a very nice result to get Lyapunov stability of a closed-loop scheme. Our stability concept goes in this direction. Moreover, it is in a sense the maximal criterion that a positive definite function \( V \) may satisfy to prove stability of the dynamical system in (8.32).

The motivation for these two stability concepts are mainly the following:

- First, the idea is to analyze the stability of the transition phase as an independent phase, using tools adapted to discrete dynamics. The only constraint is boundedness of the state on \( I_k \) and \( \| x(t^+_k) \| \leq \gamma \| x(t_k) \| \) where \( 0 < \gamma < +\infty \). Note that stabilization on \( \Omega_k \) is easily obtained using controllers developed for smooth phases. Hence this stability concept allows for instance to analyze the transition phase stabilization first, and then the smooth phases stabilization. As we indicated in subsection 8.2.2, the framework developed in this chapter could be applied to manipulators that strike dynamical environments, taking into account possible rigid body shocks at the contact. Some works in this direction can be found for instance (concerning stabilization of \( \Omega_{2k+1} \)) in [127]. The major problem with such environments would be to design a stable transition phase. It is clear that due to the motion that the obstacle undergoes between impacts, stabilization of \( I_k \) (without environment's state vector measurements) is much more difficult than with a fixed environment. We do not analyze such systems here, but content ourselves with fixed environments.

- Secondly, given a sequence of intervals \( \Omega_k \) and \( I_k \), such stability concept possesses the advantages of avoiding the difficulties discussed in chapter 7, section 7.1, and recalled in the introduction of this section: indeed we do not incorporate directly the impact phase into the stability analysis. We rather treat it as an independent phase. Moreover it is close in spirit to Lyapunov stability.

\[\text{In the next sections, we shall always assume for simplicity that } G_i \text{ is the identity, i.e. one of the global transformations described above (for instance the MacClamroch and Wang transformation [324]) has been applied to the open-loop system. Then } \tilde{V} = V \text{ and } \tilde{x}_i = x.\]
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while allowing some freedom in the stabilization of the transition phases. It is also expected to provide a convenient framework for further studies, such as robustness issues, and the bad-timing in the switching strategies.

Lemma 8.2 conditions reinforce the behaviour of the trajectories during the impact phase. We have kept the same function $V$ throughout the whole task in order to get a sort of unified stability criterion. This seems to be the stability criterion the closest to classical Lyapunov second method. Other tools based on stability analysis for hybrid dynamical systems described in [63] can be adapted to this study, in particular use of several functions $V_i$'s with additional conditions to be verified by the $V_i$'s at the switching times. The basic idea here is to attach a particular positive-definite function to each phase. We shall come back on such extensions later (see section 8.7).

The various stability criteria in lemma 8.1 are meant to apply to different classes of robotic tasks. It is also possible that other criteria can be invented to cope with other situations, see for instance section 8.7 and remark 8.32. We however limit ourselves in this study to these three ones:

- The first one in a) is the most direct extension of Lyapunov second method. It can represent the case when the task is composed of free-motion phases, separated by some impact phases.

- Case b) relaxes the conditions and is typical for tasks involving an infinity of transition phases, and where the controllers are dynamic state feedback laws.

- Case c) may be suitable when the task ends in a smooth phase (free or constraint motion) i.e. $\lambda[I] < +\infty$.

Remark 8.11 As noted in chapter 6, subsection 6.5.8, the constraints (if $m > 1$) may be attained simultaneously or separately. We may choose not to differentiate the impacts on different surfaces, but just require that conditions i, ii and iii of lemma 8.2 are satisfied, and define a codimension $m$ Poincaré section $\Sigma = \{x : f(q) = 0\}$. The problem here is that the singularity may be attained via successive rebounds on each surface $\Sigma_i$. Then at $t_k$, $f_i(q(t_k)) > 0$ for some $1 \leq i \leq m$. Hence the idea is to study $m$ Poincaré maps $P_{\Sigma,i}$ (recall we want stabilization on $f(q) = 0$, so we want an infinity of rebounds on each $\Sigma_i$), independently of the fact that some $t_k$'s may not be those corresponding to $\Sigma_i$ for some $i$. We could have numbered the impact times corresponding to each $\Sigma_i$ as $t_k^{(i)}$, and disregard $t_k^{(i)}$ in the stability analysis of $P_{\Sigma,j}$, for $j \neq i$. However except when the constraints are orthogonal, at $t_k^{(i)}$, all the velocity components may be discontinuous. Hence one must a priori take all the $t_k^{(i)}$'s into account for all the mappings $P_{\Sigma,j}$. As we shall see in section 8.6, it is even hard in this case to get weak stability. In case of orthogonal constraints, the shock dynamics are decoupled. At $t_k^{(i)}$, the velocities $\dot{q}_{\text{norm},j}$ remain continuous. It is possible, as we shall see in section 8.6, to obtain in closed-loop $n$ one-degree-of-freedom systems that evolve independently one from each other between and at the impacts.
Remark 8.12 We could have stated the concepts of weak and strong stability through an $\varepsilon - \delta$-like definition (using definitions 7.1 and 7.2), and then present lemmas 8.1 and 8.2 in the spirit of Lyapunov's second method (see for instance Vidyasagar's book [554]). In practice however, one always needs a Lyapunov-like function of the state to establish stability results. Also we could have attached directly the definition of weak stability to (closed-loop) systems as in (8.32), and that of strong stability to systems as in (8.34). Finally we have chosen the names weak and strong to emphasize that the second concept is more demanding concerning the closed-loop system's trajectories. But continuous and mixed stability would also have been suitable. This is not a very important point, only semantics.

8.5 A one degree-of-freedom example

In the following we first illustrate on a one-degree-of-freedom example how the control strategy (which encompass the switching times that define the subtasks as well as the desired trajectory definition) can influence the type of stability of the closed-loop scheme. We gradually introduce weakly stable schemes and then show how strong stability can be obtained, by modifying both the switching times and the free-motion desired trajectory. The goal of this section is mainly to show how the control strategy can be modified in various ways to comply with the stability requirements of lemmas 1 and 2.

8.5.1 Static state feedback (weakly stable task example)

Let us consider a simple one-degree-of-freedom example, where dynamical equations are given by

$$m\ddot{q} = u, \quad q \geq 0$$

(8.35)

i.e. a mass $m$ acted upon by a control force $u \in \mathbb{R}$, and restricted to move on the half-line $q \in [0, +\infty)$. Let us consider the control input

$$u(t, q, \dot{q}) = \alpha(t)u_{nc}(t, q, \dot{q}) + [1 - \alpha(t)]u_t$$

(8.36)

where

$$u_{nc}(t, q, \dot{q}) = m\ddot{q} - \lambda_1 \ddot{q} - \lambda_2 \dot{q}$$

(8.37)

$$u_t = -F_d, \quad F_d > 0$$

(8.38)

$$\alpha(t) = \begin{cases} 
1 & \text{for } t \in \Omega_{2k} \\
0 & \text{for } t \in I_k \cup \Omega_{2k+1} 
\end{cases}, \quad k \geq 0$$

(8.39)

and the system is supposed to be initialized with $q(0) > 0$, i.e. $0 \in \Omega_0$. Hence $\Omega_{2k}$ denotes free-motion phases and $\Omega_{2k+1}$ constrained-motion phases. The free motion
8.5. A ONE DEGREE-OF-FREEDOM EXAMPLE

desired trajectory $q_d(t)$ is defined as follows and is depicted in figure 8.3:

$$q_d(t) \begin{cases} 
q_d(t) \geq 0 \text{ for } t \in \Omega_k, k \geq 0 \\
q_d(t) \text{ twice-differentiable} \\
q_d(t) = q_d(t^2_k) \exp \left( -\frac{\gamma}{m} (t - T^2_k) \right) - \gamma \text{ for } t \in [T^2_k, t^2_k], k \geq 0, \gamma > 0 \\
t^1_k \text{ such that } t^1_k \leq t_j^- \text{, } |\dot{q}(t_j^-)| > \frac{2c\gamma}{m(1-e)}, q_d(t^1_k) > 0 \\
q_d(t) \equiv 0 \text{ on } \Omega_{2k+1}
\end{cases} \tag{8.40}$$

i.e. $t_j$ is a shock instant such that $|\dot{q}(t_j^-)|$ is still large enough.

We shall also consider the positive-definite function

$$V(x) = \frac{1}{2} m \dot{q}^2 + \frac{1}{2} \lambda_2 \ddot{q}^2 + c\dddot{q} = x^T P x \tag{8.41}$$

with $0 < c < \sqrt{m\lambda_2}$ and $x^T = (\ddot{q}, \dddot{q})$. The desired trajectory is therefore defined such that when a constrained phase is desired, the free-motion phase ends with an exponentially decreasing $q_d(t)$. It is supposed that the time $T^2_k < t^0_k$ is such that no collision has occurred for $t < T^2_k$, $t \in \Omega_{2k}$. Furthermore the transition phase $I_k$ is chosen to end at a finite time $t^1_j$ such that the sequence of rebounds of the mass is finished and $q_d(t^1_j) = 0$. Notice that the value of $t^1_j$ needs not to be known explicitly for the stability analysis: only its existence is needed, since both controllers on $I_k$ and $\Omega_{2k+1}$ are equal. The time $t^0_k$ is defined such that the free-motion closed-loop equation fixed point $\ddot{q} = 0, \dot{q} = 0$ is such that $q_d(t) > 0$. Hence the closed-loop equation on $\Omega_{2k}, k \geq 0$, is given by

$$m\ddot{q} + \lambda_1 \dot{q} + \lambda_2 \ddot{q} = 0 \tag{8.42}$$

$$\sigma \ddot{q}(t_k) = -(1 + e)\dot{q}(t^-_k) - (1 + e)\ddot{q}(t_k) = \sigma \ddot{q}(t_k)$$
which represents a measure differential equation with fixed point $\ddot{q} = \dot{q} = 0$, and impulsive disturbances $m(a(t)\delta(t_k))\delta(t_k) = P_q(t_k)\delta(t_k)$. Equation (8.42) can be rewritten as:

$$m\ddot{q} + \lambda_1\dot{q} + \lambda_2q = m\sum_k \sigma_{q}(t_k)\delta(t_k)$$  \hspace{1cm} (8.43)

As we discussed in chapter 7, section 7.1, if $q_d(t) < 0$, then it makes no sense to speak of Lyapunov stability for (8.42) or (8.43), because the fixed point (if any) belongs to $\partial\Phi \times \mathbb{R}$, i.e. here $q = \dot{q} = 0$. When $q_d(t) > 0$, this may be done following the tools developed in [27] [452]. The usefulness of the exponentially decaying $q_d(t)$ in (8.40) on the interval $[T^{2k}_1, T^{2k}_k]$ is to guarantee that (8.43) satisfies the requirements of lemma 8.1, as proved below. In other words, the approach phase on $[T^{2k}_1, t^k_0] \subset \Omega^{2k}$ is designed with a $q_d(t)$ such that eventual shocks do not prevent $V$ from decreasing.

Notice that due to the switching function $a(t)$ in (8.39), the transition phase dynamics are

$$m\ddot{q} = -F_q \ , \ q \geq 0$$  \hspace{1cm} (8.44)

together with the restitution rule in (8.8). Hence the sequence $\{t_k\}$ of impact times is guaranteed to exist, with a finite accumulation point $t_\infty$ (This is exactly the dynamics of the bouncing ball, see chapter 7, subsection 7.1.4). Let us analyze the variations of $V$ in (8.41):

1) On $\Omega^{2k}$ (see (8.42))

$$\dot{V} \leq -\gamma(||x||), \ \dot{x}^T = (\ddot{q}, \dot{q})$$

for suitable choice of $\lambda_1 > 0$, $\lambda_2 > 0$. Moreover on $[T^{2k}_1, T^{2k}_k] = [T^{2k}_1, t^k_0]$ one obtains

$$\sigma_V(t_k) = \frac{1}{2}m(e^2 - 1)(\dot{q}(t_k))^2 + (1 + e)\dot{q}(t_k)[m\dot{q}_d(t_k) + cq_d(t_k)]$$

$$\leq (1 + e)\dot{q}(t_k) \left[ \frac{m}{2} (e - 1)\dot{q}(t_k) - c\gamma \right] \leq 0$$  \hspace{1cm} (8.45)

for all $e \in [0, 1)$ and $q_d(t)$ as in (8.40). It is clear that with $\gamma = 0$ in (8.40), the condition $\sigma_V(t_k) \leq 0$ could be satisfied on $[T^{2k}_1, t^k_0]$ so that one could just choose $t^k_0 = t^k_1$. Note that with $\gamma = 0$, $t^k_0$ could be chosen after the collisions have stopped, i.e. one could consider all the impact phase in $\Omega^{2k}$ and no transition phase at all. In summary, $t^k_0$ could be expressed as a function of $\gamma$, with $t^k_0(\gamma = 0) = t^k_1$. Such a strategy is however feasible only if the dynamics in (8.43) with $q_d(t)$ as in (8.40) with $\gamma = 0$ guarantee convergence of $q(t)$ to zero in finite time, which has not been proved, whereas the dynamics in (8.44) do this job. In other words we need to stabilize the system on the constraint, which is a more demanding task than simply studying Lyapunov stability of the measure differential equation in (8.43).

2) On $\Omega^{2k+1}$

Then $V \equiv \dot{V} \equiv 0$, since $\ddot{q} \equiv \dot{q} \equiv 0$ (see figure 8.3).
3) On $I_k$ (see (8.44))

$$V(t_j^k) = 0 \leq V(t_j^k) = \frac{1}{2} m \left( \ddot{q}(t_j^k) \right)^2 + \frac{1}{2} \left( q(t_j^k) \right)^2 + cq(t_j^k) \ddot{q}(t_j^k) \geq 0,$$

and $V$ is uniformly bounded on $I_k$, $k \geq 0$.

Hence the conditions of lemma 8.1, a), are fulfilled and we deduce that $\ddot{q}, \dot{q} \to 0$ as $t \to +\infty, t \in \Omega_k, k \geq 0$. In this case we get even a stronger result: $\ddot{q}(t) \equiv \ddot{q}(t) \equiv 0$ for $t \in \Omega_k, k \geq 1$. This is due to the fact that since we assume that the constraint position is known ($q = 0$), once restarting the second free-motion phase $\Omega_2$, the initial conditions $\ddot{q}(T_0^2) = \ddot{q}(T_0^2) = 0$, hence perfect subsequent tracking. This is obviously an ideal situation which will not occur in practice. The assumption that the constraints $f(q) = 0$ are known is nevertheless a current assumption in basic theoretical studies [324] [607] of force/position control.

Remark 8.13 Assume that $u = u_{uc}$ for all $t \geq 0$, with $q_d < 0$ a constant signal.

Strictly speaking, there is no free-motion phase since the fixed point of the free-motion closed-loop equation does not belong to $\text{Int}(\Phi) = (0, +\infty)$. Hence this task (PD control and a constraint) reduces to $R + = -T_2 t_2 \to 0$, where $A[I_0] < +\infty$ and $A[I_0] = +\infty$ if the transition phase is stable. $\Omega_0$ is the constrained phase. First note that since the unconstrained system is globally asymptotically stable with Lyapunov function $V$ in (8.41), the sequence $\{t_k\}$ exists, because each time the system verifies $q(t) > 0$ it tends to reach the constraint. After each rebound this is true. The dynamics are similar to those of a bouncing ball with dissipation during the flight times. Let us assume (see subsection 8.5.2) that $t_\infty < +\infty$ so that one can choose $I_k$ with $t_j^k < +\infty$. Then the positive definite function $V_0$ obtained by taking $q_d = 0$ in (8.41) satisfies the requirements of lemma 8.1 (see also (8.45 which proves that $V_0(t_k) \leq 0$). One therefore sees that the conditions of lemmas 1 and 2 do not necessarily require that a switching (or discontinuous) controller be applied: this depends on the task.

Remark 8.14 The switching time $T_2^{2k} = t_0^k$ has to be computed from (8.45), insuring that the right-hand-side is negative. It may be estimated from the closed-loop dynamics in (8.42), or scheduled on-line from velocity $\dot{q}(t)$ measurements. From the fact that on $\Omega_{2k}, k \geq 1$, perfect tracking is guaranteed, no shock will occur on $\Omega_{2k}, k \geq 1$, before $t_1^k$, i.e. when $q_d(t)$ attains 0. Hence one can just choose $t_0^k = t_1^k$ starting from the second free-motion phase. On the other hand from (8.40) one sees that (8.45) is verified as long as $|\dot{q}(t_k)|$ is large enough. If $e$ is close to 1 then there may not be any such velocity. Then one has to switch to $u_4$ at $T_1^{2k}$, i.e. $t_0^k = T_1^{2k}$. Note however that $c$ and $\gamma$ can be made arbitrarily small so that one should be able to choose $t_0^k > T_1^k$. This reflects the fact that if the impacts do not dissipate enough energy, then the approach phase has to be slowed down by decreasing $c$ and $\gamma$ to still guarantee conditions of lemma 8.1, in particular negative jumps of $V$ at eventual shocks. We shall propose later a control strategy in which these problems are avoided by defining the transition phase in a different manner. Notice anyway that the designer has the freedom to choose different values of $t_0^k$ and of the desired motion velocity while still guaranteeing weak stability of the closed-loop scheme.
theoretical study merely shows that in order for a tracking controller to be Lyapunov stable when collisions occur with an obstacle, one has to modify the relative desired motion between the obstacle and the system to be controlled to guarantee (8.45).

8.5.2 Towards a strongly stable closed-loop scheme

We examine now how the transition phase controller $u_t$ and the desired trajectory influence the stability of the discrete-time mapping (the impact Poincaré map) $P_E$ during the rebounding phase. Clearly a sufficient condition for $V(t^{k+1}_E) \leq V(t^k_E)$ to be verified is to search for a control input during $I_k$ such that i) and ii) in lemma 8.2 are verified, at least for all $k \geq n$ for some finite $n \in \mathbb{N}$. We have seen that i) implies conditions on $q_d(t)$ (see (8.45)). ii) can be calculated as follows

$$
\int_{t^k}^{t^{k+1}} \dot{V}(t) dt = V(t^{k+1}_E) - V(t^k_E) = \int_{t^k}^{t^{k+1}} \{\ddot{q}[u - \ddot{q}_d] + \lambda_2 \dddot{q} + c\dddot{q} + \dddot{q}[u - \ddot{q}_d]\} dt
$$

$$
= \int_{t^k}^{t^{k+1}} \{\dot{q} + c\dot{q}\}[u - \ddot{q}_d] + c\dddot{q}^2 \} dt + \frac{\lambda_2}{2} [q^2(t^{k+1}_E) - q^2(t_k)]
$$

(8.46)

If $q_d \equiv 0$ on $(t_k, t_{k+1})$, then we get

$$
V(t^{k+1}_E) - V(t^k_E) = \int_{t_k}^{t^{k+1}} \{(\dot{q} + c\dot{q})u + c\dddot{q}^2\} dt
$$

(8.47)

One has to find out $u_t$ such that the sequence $\{t_k\}$ exists with finite accumulation point, and such that the right-hand-side in (8.47) is negative. Let us note that $u_t = -F_d$ is not suitable, as one computes that it gives

$$
V(t^{k+1}_E) - V(t^k_E) = -\frac{cF_d}{m} \int_{t_k}^{t^{k+1}} q(t) dt + c \int_{t_k}^{t^{k+1}} q^2(t) dt
$$

(8.48)

The last term hampers to state the desired result. Therefore such a simple controller (the dynamics of a bouncing ball with no dissipation during flight-times) does not allow to conclude on strong stability (although the impact Poincaré map $P_E$ is Lyapunov stable. Recall that strong stability implies Lyapunov stability of $P_E$, but the inverse is not necessarily true).

Let us consider (18):

$$
u_t(q) = -(\lambda + c)\dot{q} - F_d \quad , \quad F_d > 0, \lambda > 0 \quad (8.49)$$

Notice first that introducing (8.49) into (8.35) we obtain the transition phase closed-loop equation in (8.18)

$$
m\ddot{q} + (\lambda + c)\dot{q} = -F_d \quad (8.50)
$$

It has been experimentally shown in [268] that velocity feedback improves the system's behaviour during the transition phase. Hence the interest of considering $u_t$ in (8.49) instead of $u_t$ in (8.38).
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Similarly as in (8.44) one sees from (8.50) that the fixed point of the closed-loop equation on $I_k$ belongs to $\partial \Phi \times \mathbb{R} = \{0\} \times \mathbb{R}$. Hence

$$q(t) = \left[ \frac{F_d}{\lambda + c} + \dot{q}(t^+_k) \right] \frac{m}{\lambda + c} \left[ \exp \left( -\frac{\lambda + c}{m} (t - t^+_k) \right) - 1 \right] + \frac{F_d}{\lambda + c} (t - t^+_k) + q(t^+_k)$$

(8.51)

It follows that $q(t) \to -\infty$ as $t \to +\infty$, hence there exists a shock instant $t_k$ such that $\dot{q}$ is discontinuous at $t_k$. Then after $t_k$,

$$q(t) = \left[ \frac{F_d}{\lambda + c} + \dot{q}(t^-_k) \right] \frac{m}{\lambda + c} \left[ \exp \left( -\frac{\lambda + c}{m} (t - t^-_k) \right) - 1 \right] + \frac{F_d}{\lambda + c} (t - t^-_k)$$

(8.52)

so that another shock must occur at $t_{k+1} > t_k$. We deduce that $u_t$ in (8.49) guarantees that the system collides the constraint after the rebounds. Now the question is that since there is dissipation of energy during the flight-times, does the sequence $\{t_k\}$ possess a finite accumulation point for all $e \in [0, 1)$? This is what we discuss now.

Dissipation during flight-times and finite time stabilization

It is noted worthy that the Poincaré map associated to (8.50) and the section $q = 0$ cannot be calculated explicitly. Indeed this would require the calculation of the impact times $t_k$, which are given by $q(t_k) = 0$, i.e. the transcendental algebraic equation:

$$\left[ \frac{F_d}{\lambda + c} + \dot{q}(t^+_k) \right] \frac{m}{\lambda + c} \left[ 1 - \exp \left( -\frac{\lambda + c}{m} (t_{k+1} - t_k) \right) \right] - \frac{F_d}{\lambda + c} (t_{k+1} - t_k) = 0$$

(8.53)

Equation (8.53) is an implicit equation for the flight-time $\Delta_{k+1} = t_{k+1} - t_k$. In the case of a bouncing ball, the dissipation is generally assumed to be zero ($\lambda + c = 0$), and the flight-times are easily calculable as $\Delta_{k+1} = \frac{E_0}{m} \dot{q}(t^+_k)$ (that can be obtained from (8.53) by taking the limit as $\lambda + c \to 0$ and eliminating the solution $\Delta_{k+1} = 0$), which allows to obtain the sequences $\{\Delta_k\}$ and $\{\dot{q}(t^+_k)\}$ explicitly.

The closed-loop system on $I_k$ and when $q_d \equiv 0$ is the classical bouncing ball dynamics with dissipation during flight times. It is easy to verify that as $\lambda + c \to 0$, the solution between impacts given in (8.52) converges towards the one obtained by taking $\lambda + c = 0$ in (8.50). This is the result of continuity of the solution of ODE's with respect to parameters (which is equivalent to that with respect to initial conditions). Now since the impact times are also continuous functions of the parameters, it follows that for $\lambda + c$ arbitrarily close to zero, the solutions are arbitrarily close to those of the bouncing ball without dissipation. However we are reasoning here in the time-scale $k$, and we have seen in chapter 1, section 1.3.2 that we must be careful with what might happen in the limit as $k \to +\infty$. Here we would like to study the behaviour of $t_\infty(\lambda + c)$ for $\lambda + c > 0$. One could be tempted to state at once that since for any finite $k$ the solutions are continuous with respect to the parameters, $t_\infty$ should share the same property.
In fact we know that \( \lim_{k \to +\infty} \sum_{t=0}^{k} \lim_{(\lambda + c) \to 0} \Delta_k (\lambda + c) = t_\infty(0) < +\infty \). But is this expression equal to \( \lim_{(\lambda + c) \to 0} \lim_{k \to +\infty} \sum_{t=0}^{k} \Delta_k (\lambda + c) \)? If, for instance, the functions \( t_k(\lambda + c) \) converge uniformly to \( t_\infty(\lambda + c) \) on \([0, a]\) for any \( a > 0 \), then \( t_\infty(\lambda + c) \) is continuous and the result follows [464] theorems 7.11 and 7.12.

We have

\[
\dot{q}(t_{k+1}) = - \left( \frac{F_d}{\lambda + c} + \dot{q}_k \right) \exp \left( -\frac{\lambda + c}{m} \Delta_{k+1} \right) + \frac{F_d}{\lambda + c} \tag{8.54}
\]

and still, if \( \lambda + c = 0 \), it follows that \( t_k \to b - \frac{e^{k}}{\lambda} \). Notice that the introduction of dissipativity yields a discrete-time nonlinear system of a form that is a particular case of (7.42):

\[
\begin{align*}
\{ 
& f(\lambda + c, \dot{q}_{k-1}, \Delta_k) = 0 \\
& \dot{q}_k = g(\lambda + c, \dot{q}_{k-1}, \Delta_k)
\end{align*}
\tag{8.55}
\]

and we precisely want to study the behaviour of the solution of this system. As shown in the next subsection, it is easily proved that \( \dot{q}_k \leq e^{k+1} \dot{q}_0 \). \( u_t \) in (8.49) guarantees exponential convergence of the velocity to zero. Introducing the first equation of (8.55) in the second we obtain a nonlinear mapping relating \( \dot{q}_k \) and \( \dot{q}_{k-1} \). Then the iteration of this mapping yields the flight times expression, and \( t_\infty(\lambda + c) \). It is easy to see that there are two solutions to the nonlinear algebraic equation in (8.53): \( \Delta_{k+1} = 0 \) and \( \Delta_{k+1} = \Delta_{k+1}^{\text{max}} > 0 \). We would like to characterize \( \Delta_{k+1} = \Delta_{k+1}^{\text{max}} \). Let us note that the Taylor-Lagrange expansion of the left-hand-side of (8.53) around \( \lambda + c = 0 \) is equal to:

\[
\Delta_{k+1}^2 \left( \frac{F_d}{2m} + \frac{\lambda + c}{2m} \dot{q}_k^2 \right) - \Delta_{k+1} \dot{q}_k = \left( \frac{mF_d}{(\lambda + c)^2} + \frac{m}{\lambda + c} \dot{q}_k \right) \mathcal{O} \left((\lambda + c)^3\right) \tag{8.56}
\]

The problem is that \( \mathcal{O}((\lambda + c)^3) \) is a function of \( \Delta_{k+1} \). So we cannot use (8.56) to solve for \( \Delta_{k+1} \) as the solution of a linear second order algebraic equation.

In conclusion to this part, we believe that even if the controller in (8.49) has not yet been shown to guarantee finite time stabilization in theory, it is worth incorporating it in a control strategy, for practical purposes. Note however that one can always choose to switch to \( u_t \) in (8.38) to guarantee \( t_\infty < +\infty \). In the following we shall assume for convenience that \( u_t \) in (8.49) does the job, anyway.

**Remark 8.15** The stabilization of a system in finite time on a surface is a particular case of what is sometimes called the *slippage motion* [144] [145]. Slippage motions denote those trajectories which attain the surface, rebound infinitely often and then leaves it, i.e. reenter \( \text{Int}(\Phi) \) with no subsequent collisions. Feigin [145] provides sufficient conditions for the existence of such slippage motion in the case of a system composed of two subsystems, and a constraint of the form \( q_1 - q_2 \geq 0 \). Approximating methods for the calculation of slipping state regions and the existence regions of periodic motions with a slipping state segment in parameter space
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are studied in [144]. Of particular interest if the fact that the authors derive series approximating the total duration of the slippage motion (i.e. \( h = \sum_{k \geq 0} \Delta k \)), and of the ratio \( \frac{\Delta k}{k} \). This might be used to investigate the finite time stabilization when \( u_t \) is given by (8.49). Infinite rebounds within a finite time is also sometimes called quasi-plastic collisions [391] [393]. The system’s equations are transformed via a particular expansion of the position at \( t_k \) to obtain an approximating form of (7.42). Then an approximation of the total rebounds period length \( \sum_{k \geq 0} \Delta_k \) is derived, and conditions under which this quantity is bounded are studied (i.e. when does the quasi-plastic impact occur?). This is used in [295] to study the motion of a particle bouncing on an inclined with fixed angle but tangentially and normally moving plane, and in [392] to investigate the motion of a free system consisting of two bodies coupled by a spring and striking a fixed rigid plane.

Local (in terms of the approach velocity) expressions of the flight-time by approximating the unilateral constraint by a compliant obstacle have been obtained in the literature, see e.g. [292]. Then results on closeness of the solutions of the approximating problem to solutions of the limit rigid one (we already analyzed this topic in chapter 2) can be used to derive a closed form of the flight-times, as \( \Delta_k = c\dot{q}(t_k^-) + o(q(t_k^-)^2) \), for some constant \( c \). This is for instance used in [237] to study the C-bifurcation through approximating problems.

Let us finally note that all those approximation techniques aim in fact at solving the nonlinear set of discrete-time equations in (7.42) that represent the impact Poincaré map associated to the system.

Stability of the Poincaré impact mapping

The impact Poincaré map \( P_x \) fixed point stability is provable as shown next although \( P_x \) is not calculable explicitly. Here \( x = (\dot{q}, q) \), \( P: (0, \dot{q}(t_k^+)) \mapsto \left( \begin{array}{c} 0 \\ \dot{q}(t_k^+) \end{array} \right) \) and \( P_x: \dot{q}(t_k^+) \mapsto \dot{q}(t_{k+1}^+) \). Also \( V \) in (8.41) with \( q_d \equiv 0 \) reduces to \( V = \frac{1}{2}m\dot{q}^2 \), \( \bar{x} = x \) and \( \bar{q} = q \).

Indeed one gets from (8.47) and (8.49)

\[
V(t_{k+1}^-) - V(t_k^+) = - \int_{t_k^+}^{t_{k+1}^+} \lambda q^2(t) dt - F_d [q(t)]_{t_k^+}^{t_{k+1}^+} - c\frac{\lambda - c}{2} [q^2(t)]_{t_k}^{t_k+1} - cF_d \int_{t_k}^{t_{k+1}^+} q(t) dt \\
\leq -\lambda \int_{t_k^+}^{t_{k+1}^+} q^2(t) dt \leq 0
\]

(8.57)
since \( q(t) > 0 \) on \( (t_k, t_{k+1}) \) and \( F_d > 0, c > 0, q(t_k) = 0 \). Hence condition ii) in lemma 8.2 is verified if \( u = u_t \) as in (8.49). Now since \( q_d \equiv 0 \) it follows from (8.45) that i) in lemma 8.2 is verified also. Moreover notice that \( V_\Sigma(\dot{q}) = \frac{1}{2}m\dot{q}^2 \) so that \( V_\Sigma(0) = 0 \). We conclude following [164] that the Poincaré map \( P_x: \dot{q}(t_k^+) \mapsto \dot{q}(t_{k+1}^+) \) associated to the system in (8.35) in closed-loop with (8.49) has a Lyapunov stable fixed point \( \dot{q}^* = 0 \), with Lyapunov function \( V_\Sigma(\dot{q}) \). We have thus transformed the continuous-time Lyapunov function (for free-motion phases) \( V \) in (8.41) into a reduced order discrete-time Lyapunov function (for impact or transition phases) \( V_\Sigma \).
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Figure 8.4: Continuous desired trajectory.

Remark 8.16 Can we deduce asymptotic stability of $P_z$ from (8.57)? The whole point is to be able to extend Krakovskii-La Salle’s theorem [554] to this case. In other words, are we allowed to analyze the invariant set in the set $S = \{ q(t_k) : \int_{t_k}^{t_{k+1}} \dot{q}^2(t)dt = 0 \}$? Intuitively one may state that if $\int_{t_k}^{t_{k+1}} \dot{q}^2(t)dt = 0$ then necessarily $\dot{q}(t_k) = 0$ (because $\dot{q} \equiv 0$ on $(t_k, t_{k+1})$). Consequently $q(t) \equiv \dot{q}(t) \equiv 0$ for all $t > t_k$. Therefore the largest invariant set in $S$ is reduced to the origin, which should be Lyapunov asymptotically stable. Another path to investigate is obviously to use the nonsmooth Zhuravlev-Ivanov change of coordinates and use Lyapunov stability results for systems with discontinuous vector-fields.

In relationship with the finite-time stabilization problem, a path to investigate is first to prove the asymptotic stability of the Poincaré map (which guarantees that the velocity at impacts decreases to zero and thus attains any strictly positive value, however small, in a finite time), and then use a local analysis like the ones in [391] or [574] to prove that the sequence $\{t_k\}$ has a finite accumulation. In particular [391] studies conditions for the existence of quasi-plastic impacts (i.e. sequences of collisions that end in finite time) in general $n$-degree-of-freedom systems. When applied to the closed-loop system in (8.50), the results in [391] (see equations (2.8) in this reference) yield $\Delta_k = \varepsilon \frac{2}{F_d} e^k$ and $\dot{q}_k = \varepsilon e^k$, where $\varepsilon$ is a small constant. One is thus tempted to conclude that after a while, when $\Delta_k$ is small enough, finite time stabilization is guaranteed. Notice that the obtained expressions are nothing else but those for the bouncing ball on a fixed plane in (7.19). Wang [574] linearizes the system in (7.42) (or (8.55)) around the fixed point $t^* = t_\infty$, $\dot{q}_k = 0$ (simply taking the Taylor expansion of the functions $f$ and $g$ around the fixed point and neglecting terms of order higher than 1). But it is not guaranteed that the analysis in [574] applies directly to the system (8.55) due to the presence of the dissipation coefficient $\lambda + c$.

It remains however to determine how to choose $q_d(t)$ so as to get $q_d \equiv 0$ on $I_k$. The choice in (8.40) implies that $q_d(t_k) = 0$ for a certain time $t_k^*$ and on $(t_k^*, t_k^* + \delta)$, $q_d(t) \leq 0$, and $q_d(t_k^* + \delta) = 0$, $\delta > 0$ (see figure 8.3). It is clear that $\delta$ can be made
8.5. A ONE DEGREE-OF-FREEDOM EXAMPLE

Another choice of \( q_d(t) \) in (8.40) might be a discontinuous \( \dot{q}_d(t) \) such that at \( t^k \) (recall that \( \dot{q}_d(t^k) = 0 \), \( \dot{q}_d(t) \) jumps to zero and \( q_d \equiv 0 \) on \((t^k_1, T_{2k+1}^f)\), see figure 8.4. This implies to redefine \( u_{nc} \) in (8.37) at \( t = t^k \) (by simply disregarding the Dirac measure in \( \dot{q}_d \) at \( t^k \) since the value of \( u_{nc} \) on \([t^k_1, T_{2k+1}^f] \subset [t^k_0, T_{2k+1}^f] = I_k \cup \Omega_{2k+1} \) is useful for the analysis only but is not applied, see (8.39)). The jump in \( \dot{q}_d \) at \( t^k \) will however introduce a jump in \( V \) calculated as

\[
\sigma_V(t^k) = \frac{m}{2}(e^2 - 1) \left( \dot{q}(t^k) \right)^2 - \frac{m}{2} \left( \dot{q}_d(t^k) \right)^2 + \dot{q}_d(t^k)q(t^k) \]

(8.58)

if it is assumed that the jump occurs in \( \dot{q}_d(t) \) at the same time as a collision, i.e. \( q(t^k) = 0 \). From (8.58), \( \sigma_V(t^k) \) may be negative for sufficiently large \( \dot{q}_d(t^k) \) and sufficiently small \( \dot{q}(t^k) \). If the jump in \( \dot{q}_d \) occurs outside a collision, then

\[
\sigma_V(t^k) = -\frac{m}{2} \left( \dot{q}_d(t^k) \right)^2 + m\dot{q}_d(t^k)\dot{q}_d(t^k) + 2c\ddot{q}(t^k)\dot{q}(t^k) \]

(8.59)

whose sign may be negative for sufficiently large \( \dot{q}_d(t^k) \) and sufficiently small \( \dot{q}(t^k) \) and \( q \). Note from (8.58) that \( \sigma_V(t^k) = \frac{m}{2}e^2 \left( q_0(t^k) \right)^2 \geq 0 \) in this case. But the subsequent shocks will be such that the conditions 1) through iii) of lemma 8.2 will be verified.

Therefore the control strategy

\[
u(t) = \alpha(t)u_{nc}(t, q, \dot{q}) + [1 - \alpha(t)]u_t(\dot{q})\]

(8.60)

with \( u_{nc}, \alpha(t) \) and \( u_t \) defined in (8.37) (8.39) and (8.49) respectively, and \( q_d(t) \) as above (see figure 8.4), allows us to obtain the conditions of lemma 8.1, a) verified and the conditions of lemma 8.2 verified on \((t^k_1, t^k_f)\), for suitable feedback gains choices and restitution coefficient \( e \in [0, 1) \), with \( q_d(t) \geq 0 \) on \( \Omega_{2k} \) and \( V(x^*) = 0 \) on \([t^k_1, t^k_f] \subset I_k \),

\[
x^* = \left( \begin{array}{c} q^* = 0 \\ \dot{q}^* = 0 \end{array} \right) .
\]

Although the conditions of lemma 8.2 for strong stability are not completely fulfilled with this strategy, it however reinforces the "matching" between the function \( V \) and the controller on \( I_k \), compared to the previous controller in subsection 8.5.1. We thus have proved the following

Claim 8.1 The system in (8.35) in closed-loop with the controllers in (8.36) through (8.40) or in (8.36) (8.37) (8.39) (8.49) and desired trajectory in figure 8.4, is weakly stable (in the sense of lemma 8.1).

Remark 8.17 About the choice of the desired trajectory \( q_d(t) \): it follows from either the conditions of lemma 8.1 or of lemma 8.2 that one must have \( q_d(t) = 0 \) on \( \Omega_{2k+1} \) (constrained phases) and on \( I_k \). This is needed on one hand to satisfy \( V(t^k_f) \leq V(t^k_0) \): if \( q_d(t) \neq 0 \), then \( V(t^k_0) = \frac{1}{2}m \left( \dot{q}_d(t) \right)^2 + \frac{1}{2}\lambda_2 \left( q_d(t) \right)^2 + c\dot{q}_d(t)q_d(t) > 0 \).

Now for all \( \varepsilon > 0 \), for all \( \eta > 0 \), there exists \( T \in \Omega_{2k} \), such that for all \( t \geq T, t \in \Omega_{2k} \),
and for all \( \|x(T^2_k)\| \leq \eta, V(t) < \varepsilon \). In other words the system may remain long enough in the free motion phase \( \Omega_{2k} \) so that the required inequality in lemma 8.1 may never be satisfied. On the other hand application of Lyapunov techniques to study the stability of the Poincaré map fixed point \( x^{\ast}_k \) require \( V_k(x^{\ast}_k) = 0 \) [164]. This result may appear as a consequence of the choice of the positive-definite function \( V \) in (8.41). Notice nevertheless that in order to guarantee negative-definiteness of \( V \) on \( \Omega_{2k} \), this choice is the only possible one. Moreover the fact that the desired trajectory is consistent with the constraints is a logical feature. Note also that we have chosen \( q_d(t) \) in (8.40) twice differentiable, whereas it is only piecewise twice differentiable in the second strategy. From a general point of view, \( q_d(t) \) might be chosen in \( RCLBV \), as long as the free-motion controller \( u_{nc} \) is not used. In the next subsection we will describe another choice for \( q_d(t) \) slightly different from those in this section so that we get strong stability. Finally we could have defined \( q_d(t) \) that smoothly converges to zero in finite time, with \( q_d(t) \leq 0 \) for all \( t \geq 0 \), and applying \( u_{nc} \) as long as \( q_d > 0 \). From a practical point of view, this is not a good choice since it means that the robot has to slow down and attain the constraint with a zero velocity on \( \Omega_{2k}, k \geq 1 \). Such a strategy is quite time-consuming. For the sake of generality of the analysis in this paper we consider less smooth strategies which involve shocks between the robot’s tip (a simple mass for the moment) and the obstacle. Note that if the constraint position is not well-known, then anyway impacts are likely to occur. It is therefore much more interesting to consider collisions in a first theoretical framework. Moreover impacts may be desired in certain robotic tasks. Our goal in this study is therefore not to derive conditions guaranteeing that a controller exists such that no collisions occur in the system. If this can be done, the stability framework can nevertheless still be used. It is clear that a necessary and sufficient condition for no impact to occur is that \( \dot{q}_{\text{norm}}(t_k^\ast) = 0 \) in (8.9).

**Remark 8.18** Notice that on \( I_k \) the value of \( q_d \) is immaterial since it is generally not used in the controller \( u_t \). Hence setting \( q_d \) to zero during \( I_k \) aims only at adapting the function \( V \) to obtain a suitable Lyapunov function \( V_{z,1} \) for the mapping \( P_{z,1} \). This is however equivalent to defining an overall desired trajectory \( q^*_d \) with

\[
q^*_d = \begin{cases} 
q_d & \text{if } t \in \Omega_{2k} \\
0 & \text{if } t \in I_k \cup \Omega_{2k+1}
\end{cases}
\]  

We suppose that quasi-coordinates are used to define the constraints, i.e. they are of the form \( q_i \geq 0 \) for some \( i \). The two signals are depicted in figure 8.5 for a planar case. Then one can incorporate \( q^*_d \) into \( V \) by defining the tracking error as \( \bar{q} = q - q^*_d \). In that case \( q_d \) would not have to be compatible with the constraints, but \( q^*_d \) would. Anyway the inequality \( V(t_f^\ast) \leq V(t_0^\ast) \) has to be satisfied for weak stability, which implies some restrictions on \( q_d \) in relationship with the lengths of \( I_k \) and \( \Omega_{2k+1} \). Both approaches (say philosophies) are clearly the same, this is just a matter of convenience. The signal \( q^*_d \) has just to be taken equal to the \( q_d \) we defined above, and the "new" \( q_d \) coincides with the "ancient" one on phases \( I_k \) and \( \Omega_{2k+1} \) as...
above, and the "new" \( q_d \) coincides with the "ancient" one on phases \( I_k \) and \( \Omega_{2k+1} \) as defined in (8.61). One just has to recall that in any case, the Lyapunov-like function \( V \) has to be a positive-definite function of the closed-loop state vector \( x \), and that \( x = 0 \) must correspond to the closed-loop fixed point. It is note worthy that in the case of a compliant environment, then all the dynamics may be formulated as a pure motion problem (since the interaction force and the obstacle deformation are proportional), so that the whole stability analysis may be led with a unique Lyapunov function of the motion variables, see chapter 7, section 7.2. When the stiffness of the environment is known, one can adapt the desired motion to obtain the desired interactin force (see e.g. [360]). When it is unknown, things complicate since the stiffness has to be directly or indirectly estimated on-line [90] [594] [92].

Remark 8.19 The switching strategies we have considered are of exogeneous nature (or open-loop, or controlled switching [63]). One can choose to switch from one controller to another one from event-based conditions [342]: for instance set \( \alpha(t) \) to 0 at \( t_k^0 \), when \( g(t_k^0) = \mu, \mu > 0 \), and for all \( t \in [t_k^0, T_f^{2k+1}] \) for some \( T_f^{2k+1} \) to be chosen (it represents the time one desires that the robot pushes on the obstacle). In this case there must be a "memory" in the system, with an associated finite automaton. In the language of [63], this is an autonomous switching. If one chooses to set \( \alpha(\cdot) \) to 0 when \( q \) attains \( \mu \), \( q \) decreasing, and to 1 when \( q \) attains \( \mu \), \( q \) increasing, then \( \alpha(\cdot) \) is to be considered as a function of \( q \). This is an autonomous switching, without any automaton associated to it. Hence \( u \) is no longer Lipschitz continuous in \( q \). This may create existential problems for the closed-loop system solutions. As we saw in chapter 2, problem 2.2, the most general result concerning existence of solution for systems with unilateral constraints in [416] theorem 2, requires such Lipschitz continuity. Stability can nevertheless be attacked via the tools developed here, disregarding existential results. Furthermore we have presented the control strategy in such a way that the switching times are chosen from the desired trajectory. The contrary point of view could have been taken, i.e. the desired trajectory has to be defined from the switching between the different subtasks. Also some robustness purposes may lead to the choice of different switching strategies, see remark 8.29.
8.5.3 Dynamic state feedback

We now investigate the case when dynamic state feedback controllers are applied on $\Omega_k$. Let us consider the control input:

$$u(t, q, \dot{q}, z_1, z_2) = \alpha_1(t)u_{nc}(t, q, \dot{q}, z_1) + \alpha_2(t)u_c(\dot{q}) + \alpha_3(t)u_c(z_2)$$

(8.62)

where $q_d(t)$ is depicted in figure 8.6, and

$$u_{nc}(t, q, \dot{q}, z_1) = m\ddot{q}_d - \lambda_1\dot{q} - \lambda_2\ddot{q} - \lambda_3 z_1$$

(8.63)

$$\dot{z}_1 = \begin{cases} \ddot{q}(t) & \text{on } \Omega_{2k} \\ h(t) & \text{on } I_k \\ 0 & \text{on } \Omega_{2k+1} \end{cases}$$

(8.64)

with $h(t)$ such that $z_1(t^k_0 + \delta) = 0$ for some $\delta > 0$, $t^k_0 + \delta \leq t^j_f$, and such that $z_1(t)$ is smooth enough.

$$u_c(\dot{q}) = -\lambda\dot{q} - F_d$$

(8.65)

$\lambda > 0$ will be defined later.

$$u_c(z_2) = -F_d + z_2$$

(8.66)

$$\dot{z}_2 = \begin{cases} F - F_d & \text{on } \Omega_{2k+1} \\ 0 & \text{on } I_k \\ 0 & \text{on } \Omega_{2k} \end{cases}$$

(8.67)

with $g(t)$ such that $z_2(t^{2k+1}_f + \delta) = 0$ for some $\delta > 0$, $t^{2k+1}_f + \delta \leq t^{2k+2}_f$, and $z_2(t)$ smooth enough. $F$ is the force exerted by the constraint on the mass, and

$$\alpha_1(t) = \begin{cases} 1 & \text{if } t \in \Omega_{2k} \\ 0 & \text{otherwise} \end{cases}, \quad \alpha_2(t) = \begin{cases} 1 & \text{if } t \in I_k \\ 0 & \text{otherwise} \end{cases}, \quad \alpha_3(t) = \begin{cases} 1 & \text{if } t \in \Omega_{2k+1} \\ 0 & \text{otherwise} \end{cases}$$

(8.68)

Let us consider the positive-definite function

$$V(x) = \zeta^T P \zeta + \frac{1}{2} z_2^2 = x^T \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} x$$

(8.69)

with $\zeta = (z_1, \dot{q}, \ddot{q})$, $x^T = (\zeta^T, z_2) = (\dot{q}, \ddot{q}, z_2^T)$, $P > 0$ is the solution of $A^T P + PA = -Q$, $Q > 0$, where $\dot{x} = Ax$ is the state-space representation of $m\ddot{q} = u_{nc}(t, q, \dot{q}, z_1)$, $A$ Hurwitz for a suitable choice of $\lambda_1, \lambda_2, \lambda_3 > 0$. 
Remark 8.20 Notice that we allow for interaction force measurements only on $\Omega_{2k+1}$, not during the impact phase. This means that if one desires to study the rebounds phase with the integral force feedback (for instance for a bad timing of the switching), then only $-F_d$ should be taken into account in $z_2$.

From (8.62) to (8.69) one obtains the following

1) On $\Omega_{2k}$

$$\dot{V} = -\zeta^T Q \zeta$$ \hspace{1cm} (8.70)

for $t \neq t_k$.

The eventual jumps in $V$ are generically calculated as

$$\sigma_V(t_k) = \zeta^T(t_k^+) P \zeta(t_k^+) - \zeta^T(t_k^-) P \zeta(t_k^-)$$

$$= (e^2 - 1)p_{22} \left( \dot{q}(t_k^-) \right)^2 + 2[p_{22} \dot{q}_d(t_k^-) + p_{12} q_d(t) - p_{23} z_1(t_k^-)](1 + e)\dot{q}(t_k^-)$$

and $z_2(t) = z_2(t_{2k-1})$.

2) On $\Omega_{2k+1}$

If $q_d \equiv 0$, since $q(t) = \dot{q}(t) \equiv 0$, and since $z_1(t) = 0$ then $\zeta^T = (0, 0, 0)$ and

$$\dot{V} = -z_2^2(t) \leq 0$$ \hspace{1cm} (8.72)

3) On $I_k$

Figure 8.6: Twice differentiable desired trajectory (strong stability).
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\[ z_1(t) = z_1(t^*_{k}) + \int_{t^*_{k}}^{t} h(\tau)d\tau, \quad z_2(t) = z_2(T_{f}^{2k-1}). \]

Assume that \( q_d(t) \equiv 0. \) Then

\[
V(t_{k+1}^-) - V(t_{k}^+) = \int_{t_{k}}^{t_{k+1}} \dot{V}(t)dt
\]

\[
= \int_{t_{k}}^{t_{k+1}} \left\{ p_{11}q(t)\dot{q}(t) + p_{12}(\dot{q}(t))^{2} + p_{13}z_1(t)\dot{q}(t) + \frac{U(t)}{m} (p_{12}q(t) + p_{22}\dot{q}(t) + p_{23}z_1(t)) \right\} dt
\]

\[
= p_{12} \int_{t_{k}}^{t_{k+1}} \dot{q}(t)^{2}dt + \frac{1}{m} \int_{t_{k}}^{t_{k+1}} (p_{12}q(t) + p_{22}\dot{q}(t) + p_{23}z_1(t)) \dot{u}(t)dt
\]

(8.73)

Assume further that \( \delta \) in (8.64) is chosen sufficiently small so that \( t^*_{k} + \delta < t_0 \), where \( t_0 \) is the first impact time for the transition phase \( I_k \). Let us choose also \( \lambda = \frac{p_{22}}{p_{22}} + \kappa \), \( \kappa > 0 \), in (8.65) so that

\[
V(t_{k+1}^-) - V(t_{k}^+) = -\kappa p_{22} \int_{t_{k}}^{t_{k+1}} \dot{q}(t)^{2}dt - F_d p_{12} \int_{t_{k}}^{t_{k+1}} q(t)dt
\]

(8.74)

From (8.71) and under the condition that \( \delta \) is small enough (see figure 8.6) we also get \( \sigma_V(t_{k}) \leq 0 \) on \( I_k \) (recall we have assumed \( q_d \equiv 0 \)). Hence \( V \) in (8.69) verifies on \([t^*_{0} + \delta, t^*_{f}]\) conditions i)-iii) of lemma 8.2. \( P_\Sigma \) is Lyapunov stable with \( V_\Sigma \). Indeed on that period \( V(t_{k+1}^-) - V(t_{k}^+) \leq 0 \) provided \( p_{12} \geq 0 \), from (8.74) and (8.71). In other words we analyze the system on the subspace \( z_1 = q = q_d = 0 \) and the restriction \( V_\Sigma \) of \( V \) to it verifies the requirements for showing stability of the fixed point of the Poincaré impact map \( P_\Sigma \).

It remains however to determine the switching times between the different phases. From (8.71) it appears difficult to guarantee \( \sigma_V(t_{k}) \leq 0 \) during the free-motion phase. Hence it is preferable to define this time \( t^*_{k} \) before any discontinuity has occurred in the velocity, i.e. \( t^*_{k} = T_{f}^{2k} \) on figure 8.3. The input in (8.65) will guarantee stabilization in finite time on \( q = 0 \), and \( V \) in (8.69) will evolve according to the above considerations. Notice that \( V(t_{f}^+) = 0 \) and \( V(t_{0}^+) \geq 0 \), while \( V(t) < +\infty \) on \( I_k, \ k \geq 0 \). Similarly to the static state feedback feedback case, perfect tracking is obtained on \( \Omega_{2k}, \ k \geq 1 \), since \( x(T_{f}^{2k}) = 0 \). Notice that this time, since it is almost impossible to assure negative jumps of \( V \) at \( t_k \) when \( q_d(t) > 0 \), it is not necessary to define the "approach" desired trajectory as in (8.40), see also figures 8.3 and 8.4. Contrarily to the static feedback case, the approach period \([t^*_{f} - \delta, t^*_{f}]\) on figure 8.3, where \( q_d(t) \) decreases exponentially at the rate \(-\frac{\kappa}{m}\), see (8.40)), will be considered to belong to \( I_k \), not to \( \Omega_{2k} \), see figure 8.6. In other words the controller \( u_t \) is applied sooner than in the foregoing strategies. But anyway \( q_d(t) \) has to reach \( 0 \) in finite time and fast enough (as well as \( z_1 \), i.e. \( \delta \) small enough) either to get (8.74) and \( \sigma_V(t_k) \leq 0 \) satisfied, or \( V(t_{f}^+) \leq V(t_{0}^+) \). Once \( u_t \) in (8.65) is applied, \( q(t) \) and \( \dot{q}(t) \) evolve according to (8.51) between impacts. Hence it is possible to make the desired trajectory decrease at a large enough rate such that \( q_d(t) \) attains \( 0 \) smoothly before the first shock occurs. The designer has therefore to decide when \( u_t \) has to be switched on. Then \( q_d(t) \) evolves according to figure 8.6 (see also remark 8.19).
Then (8.70) is satisfied on $\Omega_{2k}$, (8.74) and $\sigma_V(t_k) \leq 0$ are satisfied on $[t_0, t^*_k] \subset I_k$ and $V$ is continuous on $I_k - \cup_k \{t_k\}$, (8.72) is satisfied on $\Omega_{2k+1}$. Moreover $V$ is uniformly bounded on $I_k$ and $V(t^*_k) \leq V(t^*_0)$. Hence conditions of lemma 8.1, b) are satisfied, since $|z_2(t)|$ is constant on $\Omega_{2k}$ and $|x(t)|$ is constant on $\Omega_{2k+1}$, and conditions of lemma 8.2 are fulfilled also. It should be noted that since only the ideal case is studied, the addition of a force integral feedback $z_2(t)$ is not mandatory (its usefulness is for eventual practical robustness properties), and $z_2(t) \equiv 0$ for all $t \geq 0$ if $z_2(t^0) = 0$. This is due to the algebraic form of the closed-loop equation on $\Omega_{2k+1}$, $u_c + F = 0$.

We thus have proved the following

Claim 8.2 The system in (8.35) in closed-loop with (8.62)-(8.68) and $q_d(t) \in C^2[\mathbb{R}^+]$, together with the switching times as in figure 8.6, is strongly stable (in the sense of lemma 8.2).

Remark 8.21 This desired trajectory $q_d(t)$ could also have been used in the static state feedback case. However our goal in section 8.5.1 was to show that it is possible, at least during a finite time period of approach, to satisfy stability conditions for measure differential equations as described in [452] [27], in the framework of feedback control of manipulators subject to unilateral constraints.

Remark 8.22 The control strategy in (8.62) through (8.68) can be generalized to more complex dynamic controllers. One important feature is the ability to reinitialize at zero or freeze the dynamic part (here $z_1$ and $z_2$) before the respective phases occur. This is quite convenient for the subsequent analysis of lemma 8.2 conditions and of the impact discrete-time mapping $P_x$, which becomes simple if all the variables (but $q$) are set to zero. Similarly as for the static case, it appears necessary to define $q_d(t)$ compatible with the constraints either to obtain lemma 8.1 or lemma 8.2 conditions fulfilled.

Remark 8.23 The controllers in (8.36) and (8.60) are discontinuous (measurable) in $t$ and Lipschitz continuous in the state variables. Hence existence of the closed-loop solutions is guaranteed for all $t \geq 0$ from [416] theorem 2, with the modifications we discussed in subsection 8.3.4 (in particular Carathéodory-like conditions). However no existing theoretical study allows us to treat the case of dynamical feedback. Once again we have to conjecture existence of solutions with velocity in $RCLBV$. Note that any dynamical part of the closed-loop state has at least the regularity of the integral of the velocity, since dynamical feedback involves at least one integrator. An open problem is therefore to prove existence of solutions $(q(t), z(t)$ continuous,
\( \dot{q}(t) \) in \( RCLBV \) for the following dynamical problem:

\[
\begin{align*}
\dot{q}(t) &= g(t, q, \dot{q}, z) \\
H &\geq f(q) \geq 0 \\
\dot{q}_{\text{norm}}(t_k^+) &= -e\dot{q}_{\text{norm}}(t_k^-) \\
0 &\leq e \leq 1
\end{align*}
\]

The functions \( g \) and \( Z \) satisfy the Carathéodory conditions. (8.75)

### 8.6 \( n \)-degree-of-freedom rigid manipulators

In this section we discuss the extension of the one-degree-of-freedom case to \( n \)-degree-of-freedom manipulators. In particular we highlight the consequences of a generalized restitution rule, through the generalized velocity transformation in (8.9). It is shown that although weak stability can in general be obtained, strong stability is more difficult to achieve when the controller is designed from generalized coordinates whose derivative does not correspond to the transformed velocity in (8.9).

#### 8.6.1 Integrable transformed velocities

Consider the rigid \( n \)-degree-of-freedom manipulator dynamics in (8.1) and the generalized velocities transformation in (8.9), and 2 orthogonal constraints \( f_1(q) \geq 0 \), \( f_2(q) \geq 0 \). Then one can write the dynamical equation as

\[
\begin{align*}
\dot{q}_{\text{norm}} - E_1 \dot{M} \dot{q} + n_q^T [C(q, \dot{q}) \dot{q} + g(q)] &= n_q^T u \\
\dot{q}_{\text{tang}} - E_2 \dot{M} \dot{q} + t_q^T [C(q, \dot{q}) \dot{q} + g(q)] &= t_q^T u
\end{align*}
\]

Equations (8.11) are true for \( t = t_k \), and \( M = \begin{bmatrix} n_q^T \\ t_q^T \end{bmatrix} M(q) \). The restitution rules are defined in (8.12), \( E_1 = [I_2; 0_{2 \times (n-2)}] \), \( E_2 = [0_{(n-2) \times 2}; I_{n-2}] \). It clearly appears from (8.76) and (8.11) why the orthogonality of the constraints allows us to treat the \( n \)-degree-of-freedom case as \( n \) one-degree-of-freedom cases. There is a coupling between the \( n \) equations in (8.76) through the Coriolis and gravity terms, but it can be compensated for via suitable feedback.

Let us apply a linearizing and decoupling control input \( u \) such that

\[
\begin{bmatrix} n_q^T \\ t_q^T \end{bmatrix} u = \begin{bmatrix} n_q^T \\ t_q^T \end{bmatrix} [C(q, \dot{q}) \dot{q} + g(q)] + \begin{bmatrix} E_1 \dot{M} \dot{q} \\ E_2 \dot{M} \dot{q} \end{bmatrix} + v \quad (8.77)
\]
8.6. N-DEGREE-OF-FREEDOM RIGID MANIPULATORS

Hence we obtain

\[
\begin{align*}
\ddot{q}_{\text{norm},1} &= v_1 \\
\ddot{q}_{\text{norm},2} &= v_2 \\
\ddot{q}_{\text{tang},1} &= v_3 \\
&\vdots \\
\ddot{q}_{\text{tang},n-2} &= v_{n-2} \\
\ddot{q}_{\text{tang},n-1} &= v_n
\end{align*}
\]  

(8.78)

Let us assume that the system under consideration is written in generalized coordinates as in remark 8.6, i.e. the unilateral constraints are simply \( q_1 \geq 0, \quad q_2 \geq 0 \) (\(^19\)), and that the generalized velocity transformation in (8.9) is defined using a basis \((n_q, t_q)\) without normalization of \( n_q,i \) in (6.2). Indeed the definition of normalized vectors \( n_q,i \) in (6.2) is convenient for instance to write down the kinetic energy loss at impacts in a quite simple form, see subsection 6.2.1. But it is not necessarily suitable for control purposes, i.e. for \( M \) to be a Jacobian. Hence we define a similar transformation as Yoshikawa in [607], but contrarily to [607] §6.2.2 we have not defined the constraints in terms of the end-effector position. Moreover the transformations in (8.9) and in [607] equations (6.43)-(6.51) are different one from each other. But we are led to make the same type of assumptions on the integrability of the transformed coordinates as in [607]. More precisely, let us do the assumption that there exist functions \( q_i(t) \) such that \( \dot{q}_i = \dot{q}_{\text{tang}} \) (Notice that \( \dot{q}_{\text{norm},i} = \zeta_i^T \dot{q} = \dot{q}_i \) so that \( q_{\text{norm},i} = q_i, \quad i = 1, 2 \)). Now let us define

\[
v_i = \alpha_1(t)u_{n_1}(t, q_i, z_{1,i}) + \alpha_2(t)u_2(q_i) + \alpha_3(t)u_3(z_{2,i})
\]

(8.79)

for \( i = 1, 2 \), where all the terms are defined as in (8.63) through (8.68), and

\[
v_i = q_{\text{tang},i} - \gamma_2 \dot{q}_{\text{tang},i} - \gamma_1 \ddot{q}_{\text{tang},i}
\]

(8.80)

for \( i = 3, \ldots, n \), where \( \gamma_1, \gamma_2 \) are suitably chosen such that the polynomial \( m \dot{s}^2 + \gamma_1 s + \gamma_2 \) is Hurwitz. Let us choose

\[
V = \zeta_1^T P_1 \zeta_1 + \zeta_2^T P_2 \zeta_2 + \frac{1}{2} z_{2,1}^2 + \frac{1}{2} z_{2,2}^2 + \sum_{k=3}^{n} \zeta_k^T P_k \zeta_k
\]

(8.81)

which can be written as

\[
V = V_1(\zeta_1, z_{2,1}) + V_2(\zeta_2, z_{2,2}) + \sum_{k=3}^{n} V_k(\zeta_k),
\]

where \( \zeta_k = (\tilde{q}_{\text{tang},k}, \dot{\tilde{q}}_{\text{tang},k}), \quad \zeta_i^T = (z_{1,i}, \tilde{q}_{\text{norm},i}, \dot{q}_{\text{norm},i}), \quad i = 1, 2 \). The \( P_i \)'s are naturally defined from the Lyapunov equations associated to each sub-closed-loop equation obtained by introducing (8.79) and (8.80) into (8.78).

It is then not difficult to show that the result of lemma 1 applies. The last term \( \sum_{k=3}^{n} V_k(\zeta_k) \) in (8.81) evolves independently of the rest of \( V \) in (8.81) and its derivative is given by \(-\zeta_k^T Q_k \zeta_k, \quad Q_k > 0, \) for all \( t \geq 0 \). Also the terms \( V_i(\zeta, z_{2,1}) \)

\(^{19}\)In other words we use once again the quasi-coordinates. Their usefulness is illustrated by many studies presented in the foregoing chapters. It has been pointed out in [25][26] that they are indeed quite suitable for the analysis of impact equations, because they alter neither the shock dynamical equations nor the variational principles for systems with unilateral constraints.
CHAPTER 8. FEEDBACK CONTROL

and \( V_2(\zeta_2, z_2, 2) \) evolve independently one from each other. The controller guarantees that the system stabilizes in finite time on each surface \( \Sigma_1, \Sigma_2 \).

Concerning strong stability, let us note that (8.78) in closed-loop with (8.79) (8.80) satisfies the requirements of lemma 2, provided both surfaces \( \Sigma_1 \) and \( \Sigma_2 \) are attained simultaneously during \( I_k \). This may be guaranteed by taking the same initial conditions and switching times in both equations governing \( q_1(t) \) and \( q_2(t) \) evolution, and with \( e_1 = e_2 \) \(^{20}\). These constraints can be relaxed if one guarantees that \( V_1 \) and \( V_2 \) decrease between impacts, or that \( V_1(t_{k+1}^-) \leq V_1(t_k^+) \), \( i = 1, 2 \). Strong stability requires that the impact Poincaré mappings \( P_{k,1} \) and \( P_{k,2} \) be Lyapunov stable. A singularity may be in general attained via a succession of simple impacts on each surface \( f_i(q) = 0 \). The criterion we have used when there is only one surface of constraint is to guarantee that \( \dot{V} \leq 0 \) between impacts, via a suitable \( u_t \), and to study conditions such that \( \sigma_V(t_k) \leq 0 \) as well. In a multiple collisions case, we can try to do the same. Several paths may be followed: a) Consider the first return maps for each surface (i.e. \( \Sigma_i = \{ f_i(q) = 0 \} \) and \( P_{z,i} \) is defined at \( t_k \)'s), or b) consider the mapping between each impact, whatever the impacting surface may be. In the first case we will have to take care of the fact that parts of the positive definite function \( V \) will jump between two impacts with the same surface, due to collisions with other surfaces. Will the jumps be negative? In the second case, we will face the difficulty of showing a result like in (8.57), where this time \( q_i \) is not necessarily 0 when an impact with \( q_j \) = 0 occurs, \( i \neq j \). Therefore the ultimate goal that consists of finding a transition phase controller \( u_t \) such that \( \dot{V} \leq 0 \) on \( (t_k, t_{k+1}^+) \), with \( i \) possibly different from \( j \), and at the same time insuring \( \sigma_V(t_k) \leq 0 \), remains open. It is likely that another concept of stability for multiple collisions has to be discovered. But before this, the dynamics of multiple collisions should certainly be more settled than what they are presently \(^{21}\).

To illustrate the difficulties involved by strong stability when several constraints are attained, we choose path a). Let us consider \( u_{nc} \) as in (8.37) and \( u_c \) as in (8.38), i.e.

\[
u_{nc}(q_i, q_i, t) = \dot{q}_{di} - \lambda_1 \dot{q}_i - \lambda_2 \dot{q}_i
\]

(8.82) for \( i = 1, 2 \), and

\[
u_c = -n_{q_i}^T \nabla q F_i(q) \lambda_{d,i} = -\nabla q F_i(q)^T M^{-1}(q) \nabla q F_i(q) \lambda_{d,i}
\]

(8.83)

\(^{20}\)But note that in certain cases, the system is such that it strikes \( \partial \Phi \) exactly at the singularity, think of the rocking block, or of bipedal locomotion (see remark 8.32). More exactly, one does the assumption that this is the case, see \([221]\). Then one may look at the system only when it attains the singularity, i.e. use a codimension 2 Poincaré section. In general however it is not possible to guarantee that the singularity is attained directly (Recall that the set of initial data such that this is the case is of zero measure \([292]\)).

\(^{21}\)Recall that in chapter 6, we have essentially focused on the definition of restitution rules when the system attains directly a singularity. This is justified by the fact that certain mechanical systems like the rocking block or the n-balls are such that indeed the singular part of the boundary \( \partial \Phi \) can be attained directly (thus although from a mathematical viewpoint one might argue that such trajectories are of zero measure in the configuration space, they are worth considering). Concerning stability results, one sees now that this in fact represents the nicest way a singularity may be attained. The case of multiple repeated simple collisions is much more involved.
with \( \lambda_d = [\lambda_{d1}, \lambda_{d2}]^T \), \( \lambda_{d,i} > 0 \), \( i = 1, 2 \). Then on \( \Omega_{2k} \), the closed-loop dynamics are 
\[
\ddot{q}_i + \lambda_1 \dot{q}_i + \lambda_2 q_i = 0
\]
and on \( \Omega_{2k+1} \) we obtain \( \lambda = \lambda_d \) (since \( q_i \equiv 0 \) and the first two equations in (8.78) become \( 0 = u_c + n_i^T \nabla_q f_i(q) \)) also \( \nabla_q f(q)^T M^{-1}(q) \nabla_q f(q) = \text{diag}(\nabla_q f_i(q)^T M^{-1}(q) \nabla_q f_i(q)) > 0 \) since \( M^{-1}(q) > 0 \) and \( \nabla_q f_i(q) \neq 0 \) by assumption. Now let us choose on \( I_k \):
\[
u_i = -\lambda_1 \dot{q}_i - \lambda_2 q_i - \lambda_{d,i}
\]
(8.84)
i = 1, 2, so that the closed-loop dynamics during flight-times are 
\[
\ddot{q}_i + \lambda_1 \dot{q}_i + \lambda_2 q_i = -\lambda_{d,i}
\]
(8.85)
It is expected that the addition of a position feedback will help in rendering \( \dot{V} \) negative between impacts. Similarly as for (8.50), it can be shown that (8.85) implies that \( \{t_k\} \) is infinite. Similarly we suppose that \( t_{\infty} < +\infty \). Now consider \( V_i \) as in (8.41), i.e.
\[
V_i(q_i, \dot{q}_i) = \frac{1}{2} \dot{q}_i^2 + \frac{1}{2} \lambda_2 q_i^2 + c \dot{q}_i \dot{q}_i
\]
(8.86)
Assume that \( q_{i,d} \equiv 0 \) on \( I_k \). Then on \( (t_k, t_{k+1}) \) one obtains 
\[
\dot{V}_i = (-\lambda_1 + c) \dot{q}_i^2 - \lambda_{d,i} \dot{q}_i - \lambda_1 c \dot{q}_i q_i - \lambda_2 c q_i^2 - \lambda_{d,i} c q_i
\]
(8.87)
from which it follows that 
\[
V_i(t_k^-) - V_i(t_k^+) = \int_{t_k}^{t_{k+1}} (-\lambda_1 + c) \dot{q}_i^2 dt - \lambda_{d,i} \int_{t_k}^{t_{k+1}} q_i^2 dt - \lambda_1 c \int_{t_k}^{t_{k+1}} q_i^2 dt - \lambda_2 c \int_{t_k}^{t_{k+1}} q_i^2 dt
\]
(8.88)
Notice that there is no reason for the second and third terms in this expression to be negative. Indeed it may happen that \( q_i(t_{k+1}) = 0 \) if an impact occurs with \( \Sigma_i \) at \( t_{k+1} \) (i.e. \( t_{k+1} = t_{k+1}^i \)), and that \( q_i(t_k) > 0 \) if the \( k \)th impact occurs with \( \Sigma_j, i \neq j \) (i.e; \( t_k = t_k^j \)). On the contrary the last term is negative since \( q_i \geq 0 \). Moreover the inequality 
\[
\dot{V}_i \leq -\gamma_i(||(q_i, \dot{q}_i)||)
\]
(8.89)
is satisfied for some class \( K \) function \( \gamma_i(\cdot) \) if \( \lambda_{d,i} = 0 \). But then finite time stabilization on the surface \( \Sigma_i \) is lost, since the equilibrium point of the smooth dynamics exactly lies on \( \partial \Phi \times \mathbb{R} \). Note that due to the orthogonality of the constraints we still get 
\[
\sigma_{V_i}(t_k) = \frac{1}{2} (\epsilon_i^2 - 1) (\dot{q}_i(t_k^i))^2 \leq 0
\]
(8.90)
for \( i = 1, 2 \), where it is understood that \( t_k = t_k^i \) in (8.90). Clearly if the constraints were not orthogonal, then at \( t_k^i \) there would be a jump in \( \dot{q}_i, i \neq j \).

A solution is to design the desired trajectories (see for instance figure 8.6 for the choice of \( q_{d,i} \)) so that the system first stabilizes on \( \Sigma_1 \) (in other words \( q_1 \) and \( \dot{q}_1 \) converge to zero in a finite time), and then on \( \Sigma_2 \) (see for instance figure 8.6 for
the choice of \( q_{d,i} \). This strategy guarantees that \( V(x) = V_1(q_1, \dot{q}_1) + V_2(q_2, \dot{q}_2) + \sum_{k=3}^n V_k(z_k) \) satisfies conditions of lemma 2. We have \( V_{\Sigma_i} = V_1(0, \dot{q}_1) + V_j + \sum_{k=3}^n V_k, \)
\( i = 1, 2, j = 1, 2, i \neq j \). The idea that consists of switching between several controllers depending on which surface is going to be attained is not good, because the surfaces may \textit{a priori} be attained in any order (i.e. how can one guarantee that \( t_k = t_k^* \) for some \( k \) and \( i \)?) \(^{22}\), and also because the finite accumulation point of \( \{t_k\} \) prevents the application of such a strategy.

\textbf{Remark 8.24} Notice that when the constraints are orthogonal (i.e. satisfy (8.4)), then \( n_{q,i} \) is tangent to \( \Sigma_j, i \neq j \). It is therefore not important to define \( t_q \) from one or the other surface to write the dynamics. Note also that despite of the fact that one chooses to stabilize the system first on \( \Sigma_1 \) and then on \( \Sigma_2 \), the orthogonality of the constraints is fundamental because it assures that \( \dot{q}_1 \) does not jump when \( \dot{q}_2 \) does, and \textit{vice-versa}.

\textbf{Remark 8.25} It is interesting to note that in proportion as the open-loop system's complexity increases, the requirements on the behaviour of \( V_i, i = 1, 2 \), become more stringent. The ultimate goal is to find out a controller \( u_t \) such that the above stated conditions (see (8.89)) are satisfied.

\textbf{Remark 8.26} Let us recall that even if the transformed velocities (or transformed generalized momentum) are integrable but are not orthogonal, then the above stability result is more difficult to be obtained. Indeed in that case a collision with surface \( \Sigma_i \) induces a jump in velocity \( \dot{q}_{\text{norm}, j}, i \neq j \). It must therefore be verified that this jump is such that it produces a negative jump in the function \( V \), and this may not be obvious. This is a problem quite similar as for non-integrable transformed velocities studied in the next subsection. In summary, we are able to extend with slight difficulties the case of a codimension one constraint to the case of codimension \( \geq 2 \), only when the constraints are orthogonal and the transformed velocities are integrable.

\textbf{Remark 8.27} Let us denote \( \ddot{q} = M\dot{q} \). Since we have assumed that there exists functions \( \dot{q}_i(t) \) with \( \ddot{q}_i(t) = \dot{q}_{\text{norm}}(t) \), we have \( \ddot{q} = Q(q) \) for some function \( Q \) with \( \frac{\partial Q}{\partial q} = M \). The above controllers assure in particular that \( \ddot{q} \rightarrow \ddot{q}_d \) asymptotically. Hence \( q \rightarrow Q^{-1}(\ddot{q}_d) \). Recall that the "natural" generalized coordinates (i.e. the links angles) from which the desired trajectory is usually planned is \( q \). One therefore has to choose \( \ddot{q}_d = Q(q_d) \). This of course requires that the inertia matrix be known, since it enters into the definition of the vectors \( t_q \), see chapter 6, section 6.2. This is the main discrepancy between this change of variables and the ones in [324] [607].

\section{Non-integrable transformed velocities}

What if \( M \) is not a Jacobian, which might be indeed the case in general? Then there exists no \( q_i(t) \) such that \( \ddot{q}_i(t) = \dot{q}_{\text{norm}} \) in (8.9). It is still possible to have \( \dot{q}_{\text{norm}} \) track

\footnote{Recall some authors even consider that in general multiple collisions are of stochastic nature.}
some desired signal \( \dot{q}_{d_{\text{ang}}}(t) \), which means that the components of \( M(q)\dot{q} \) along \( t_{q,i} \), \( 1 \leq i \leq n - 2 \) (or in other words the components of \( \dot{q} \) along \( t_{q,i} \) and calculated with the kinetic metric) track \( \dot{q}_{d_{\text{ang}}}(t) \). But the rest of the state vector (position) evolution is not clearly related to \( \dot{q}_{d_{\text{ang}}}(t) \), since \( q(t) = q(0) + \int_0^t \mathcal{M}^{-1}(q(\tau)) \left[ \dot{q}_{\text{term}} \right] \, d\tau \). In this subsection we illustrate the difficulties posed by the design of weakly or strongly stable closed-loop systems when using "classical" coordinates (see also remark 8.31 for some possible paths that should be worth following for the nonintegrable case, and therefore allow us to overcome the problems encountered in the sequel). The difficulty mainly comes from the fact that during the rebounds phase, the velocity components \( \dot{x}_2 \) will in general possess jumps at \( t_k \). Suppose that \( x_1 = 0 \). Following [324] we get in the \( X \)-coordinates the dynamical equations\(^{23}\)

\[
\begin{align*}
\dot{M}_{11}\ddot{x}_1 + \dot{M}_{12}\ddot{x}_2 + H_1(q, \dot{q}) &= T_1u + T_1P_q = T_1u + P_X \\
\dot{M}_{21}\ddot{x}_1 + \dot{M}_{22}\ddot{x}_2 + H_2(q, \dot{q}) &= T_2u
\end{align*}
\]  

(8.91)

where the various terms come from the nonsingular transformation between \( X \) and \( \begin{pmatrix} q \\ \dot{q} \end{pmatrix} \), and are defined in [324]. \( \tilde{M}_{11} > 0 \in \mathbb{R}, \tilde{M}_{12} = \tilde{M}_{21}^T \in \mathbb{R}^{1 \times (n-1)}, \) and \( \tilde{M}_{22} > 0 \in \mathbb{R}^{(n-1) \times (n-1)}. \) \( H_1(q, \dot{q}) \) and \( H_2(q, \dot{q}) \) contain the centrifugal and centripetal terms, \( T_1 \in \mathbb{R}^{1 \times n} \) and \( T_2 \in \mathbb{R}^{(n-1) \times n} \) are such that \( \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \) is full rank. \( P_X \) in (8.91) generically represents the interaction force, i.e. \( P_X = \nabla_X F(X)p_{z,1}, p_{z,1} \in \mathbb{R}^+ \) is the Lagrange multiplier associated to \( F(X) = 0 \). In a permanently constrained case \( x_1 \equiv 0 \) and the second equation in (8.91) represents the motion along the constraint, see [324]. During a collision with the constraint, application of Newton's generalized restitution rule yields \( \ddot{x}_1(t^+_k) = -e\ddot{x}_1(t^-_k) \), since \( \nabla_X F(X)^T = (1, 0, \ldots, 0) \) (in the case of a codimension 1 constraint). From (8.91) it therefore follows that

\[
\dot{M}_{21}\sigma_{x_1}(t_k) + \dot{M}_{22}\sigma_{\dot{x}_2}(t_k) = 0
\]  

(8.92)

whereas

\[
\dot{M}_{11}\sigma_{x_1}(t_k) + \dot{M}_{12}\sigma_{\dot{x}_2}(t_k) = p_{z,1}(t_k)
\]  

(8.93)

where \( p_{z,1} \geq 0 \) is the percussion vector first component. Thus the jumps in \( \dot{x}_2 \) can be calculated from the jumps of \( \dot{x}_1 \) using the algebraic dynamics at impacts, as

\[
\sigma_{\dot{x}_2}(t_k) = -\dot{M}_{22}^{-1}\dot{M}_{21}\sigma_{x_1}(t_k) = \dot{M}_{22}^{-1}\dot{M}_{21}(1 + e)\dot{x}_1(t^-_k)
\]  

(8.94)

The percussion vector is then obtained from (8.93). The only case when \( \sigma_{\dot{x}_2}(t_k) = 0 \) is when the inertia matrix in \( X \)-coordinates is block diagonal, i.e. \( \dot{M}_{12} = 0 \). If such is the case, then the algebraic impact equation in (8.93) yields \( \dot{M}_{11}\sigma_{x_1}(t_k) = p_{z,1}(t_k) \) and \( \dot{M}_{22}\sigma_{\dot{x}_2}(t_k) = 0 \). Notice that the decoupling between \( \ddot{x}_1 \) and \( \ddot{x}_2 \) is independent of the orthogonality condition (8.4) between the constraints, which concerns here \( \dot{M}_{11} \).

\(^{23}\)See remark 6.11.
when \( x_1 \in \mathbb{R}^m, m \geq 2 \). Indeed with \( \tilde{M}_{12} = \tilde{M}_{21} = 0 \), the orthogonality condition implies \( e^T \tilde{M}_{ij}^{-1} e_j = 0, i, j \in I, i \neq j \), i.e. \( \tilde{M}_{11} \) is diagonal.

We now illustrate the difficulty related to stabilization (in the sense of lemmas 1 and 2) of a complete task for the system in (8.91). In what follows \( \tilde{x}(t) = x(t) - x_d(t) \). The desired trajectories are \( x_d(t) \in [\mathbb{R}^+ \text{]} \) and \( x_{1d} \) can be chosen as in figures 8.3, 8.4 or 8.6. Let us choose the controllers as follows

- **On \( \Omega_{2k} \)**
  The controller is chosen as
  \[
  \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} u_{nc} = \tilde{M} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \tag{8.95}
  \]
  with \( v_1 \in \mathbb{R} \) and \( v_2 \in \mathbb{R}^{n-1} \). The first closed-loop equation is simply \( \ddot{x}_1 = v_1 \) and \( \ddot{x}_2 = v_2 \).
  \[
  v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \ddot{x}_d - \lambda_1 \dot{x} - \lambda_2 \dot{x} \tag{8.96}
  \]
  Hence the closed-loop equation is \( \dddot{x} + \lambda_1 \dot{x} + \lambda_2 \dot{x} = 0 \).

- **On \( \Omega_{2k+1} \)**
  The open-loop system is obtained from (8.91) by taking \( x_1 \equiv 0 \). The controller is chosen as
  \[
  T_1 u_c = \tilde{M}_{12} v_2 + H_1 - p_{x_d} \tag{8.97}
  \]
  \[
  T_2 u_c = \tilde{M}_{22} v_2 + H_2 \tag{8.98}
  \]
  where \( v_2 = \ddot{x}_{2d} - \lambda_1 \dot{x}_2 - \lambda_2 \dot{x}_2 \). Hence the closed-loop is given by \( x_1 \equiv 0, \ddot{x}_2 + \lambda_1 \dot{x}_2 + \lambda_2 \dot{x}_2 = 0 \) and \( p_{x,1} = p_{x_d} \).

The stability on \( \Omega_{2k} \) and \( \Omega_{2k+1} \) can be shown with the Lyapunov function \( V = V_1(\ddot{x}_1, \dot{x}_1) + V_2(\ddot{x}_2, \dot{x}_2) \). \( V_1 \) and \( V_2 \) are chosen as in (8.41). Clearly on \( \Omega_{2k} \) and for suitable choice of the gains \( \lambda_1, \lambda_2 \), \( \dot{V} \leq -\gamma(||(\dot{X}, \dot{\tilde{X}})||) \). On \( \Omega_{2k+1} \), if \( x_{1d} \equiv 0 \) then \( V_1 \equiv 0 \) and \( V_2 \leq -\gamma(||(\dot{x}_2, \dot{\tilde{x}}_2)||) \).

**Remark 8.28** It appears clearly from (8.78) that in the case of integrable transformed velocities, the generalized velocity transformation together with orthogonal constraints and a decoupling-linearizing controller, allows us to recast the \( n \)-degree-of-freedom \( m \)-constraints case into the framework developed in [416]. Things are not so clear now in this section 8.6.2. Indeed the state space transformation between any generalized coordinates position and velocity, say for instance \( X \) and \( \dot{X} \), and the transformed ones, i.e. \( \tilde{X} \) and \( \dot{\tilde{X}} \), is defined through the matrix \( \mathcal{M}(X) \triangleq \begin{pmatrix} I_n & 0 \\ 0 & \mathcal{M} \end{pmatrix} \). Now if we obtain \( \tilde{X} = \mathcal{M} v \), where \( v \) is an input to be defined, we get in the transformed state space \( \dot{\tilde{X}} = \mathcal{M} \dot{v} + \dot{\mathcal{M}} \tilde{X} \). Hence the decoupling is lost, and we can no longer apply
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directly the existence of solutions results of Paoli and Schatzman [416], see problem 2.2. We have to conjecture existence since the generalized force acting on the system between the collisions is not decoupled.

\[\nabla\nabla\n\]

It remains now to design the controller on \( I_k \). Let us choose \( u_t \) as in (8.95), but with \( v_1 = -1 \), and \( v_2 = \ddot{x}_2 - \lambda_1 \dot{x}_2 - \lambda_2 \dot{x}_2 \). The \( x_1 \)-dynamics are therefore those of the vertical bouncing ball, which guarantees that \( x_1 \) tends to zero in a finite time \( t_\infty \). Indeed it can be calculated that the first shock occurs at \( t_0 = \frac{1}{2} \dot{x}_1(t_0^+) + \frac{1}{2} \sqrt{\dot{x}_1^2(t_0^+) + 2a\dot{x}(t_0^+)} + t_0^+ > t_0^+ \). Then one obtains \( \dot{x}_1(t_k^+) = -e^{k+1} \dot{x}_1(t_0^-) \) and \( \Delta_k = t_k - t_{k-1} = \frac{2}{e} \dot{x}_1(t_0^-) \). From (8.94) one sees that the jumps in \( \dot{x}_2 \) also converge exponentially towards 0 in finite time (provided of course \( e < 1 \)). They are given by \( \sigma_{2|2}(t_k) = a e^{-1} M_2^{-1} M_2 \Delta_k \).

\( V_1 \) can be analyzed exactly as we did for \( V \) in (8.41). But contrarily to the integrable velocity case, \( V_2 \) is discontinuous at \( t_k \), although \( V_2 \leq -\gamma(\{\|\dot{x}_2, \ddot{x}_2\|\}) \) on \((t_k, t_{k+1}]\), for all \( k \geq 0 \). Indeed one obtains

\[
\sigma_{2|2}(t_k) = \sigma_{2|2}(t_k)^T \left( \frac{\dot{x}_2(t_k^+)}{2} + \frac{\dot{x}_2(t_k^-)}{2} + c\ddot{x}_2(t_k) - \ddot{x}_2(t_k) \right)
\]

which has no reason to be negative (see (8.94)). Recall that from weak stability conditions, one must guarantee that \( V(t_k^0) \geq V(t_k^+) \). Due to the jumps in \( V_2 \) during the rebounds phase, it is not obvious whether there exists a time \( t_f^+ < +\infty \) such that this condition is satisfied. Indeed as we noted in remark 8.17, if the system remains long enough in \( \Omega_k \), then \( V(t_k^0) \) may be arbitrarily small. But \( V_2(t_\infty) \) can be strictly positive because of some positive jumps. Notice that \( V_2(t_{k+1}) - V_2(t_k^+) = -\beta(\Delta_k^+, \lambda_1, \lambda_2, \ddot{x}_2(t_k^+), \ddot{x}_2(t_k^+)) \leq \beta_k \) for some function \( \beta \). \( \beta_k = 0 \) if \( \Delta_k = 0 \) and \( \beta_k \to +\infty \) as \( \lambda_1, \lambda_2 \to +\infty \), \( \Delta_k \geq \Delta_{\min} > 0 \). \( \Delta_k = t_k - t_{k-1} \) are the flight-times. Assume that it can be shown that there exists \( \lambda_1^* < +\infty \) and \( \lambda_2^* < +\infty \) such that \( \lambda_1 \geq \lambda_1^* \), \( \lambda_2 \geq \lambda_2^* \) implies \( |\beta_k| > |\sigma_{2|2}(t_k)| \) for all \( k \geq 0 \): this insures that despite of possible positive jumps, \( V_2 \) decreases enough between the collisions so that \( V_2(t_{k+1}) \leq V_2(t_k^+) \). Then existence of \( t_f^+ \) such that \( V(t_k^0) \geq V(t_f^+) \) is guaranteed, and its value does not depend on the value of \( V(t_k^0) \). Such conditions are evidently sufficient only to prove stability of the task.

Avoidance of collisions via dead-beat control

One possible solution to avoid these drawbacks is to design a controller such that no collision occurs during \( I_k \). Let us choose a feedback linearizing controller as in (8.95). The \( x_1 \)-closed-loop system can be written as \( \dot{x}_1 = v_1 \), or

\[
\begin{align*}
\dot{x}_{11} &= x_{12} \\
\dot{x}_{12} &= v_1
\end{align*}
\]
Let us choose \( v_1 = -a_1 x_1 - a_2 x_12 + \bar{v}_1 \), and let us denote \( X_1 = (x_{11}, x_{12})^T \), \( A = \begin{pmatrix} 0 & 1 \\ -a_1 & -a_2 \end{pmatrix} \) is a Hurwitz matrix for suitable gains, and \( B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). Hence we obtain

\[
X_1(t) = \exp \left( A(t - t_0^k) \right) X_1(t_0^k) + \int_{t_0^k}^t \exp \left( A(t - \tau) \right) B\bar{v}_1(\tau) d\tau \quad (8.101)
\]

It is possible to bring the state \( X_1 \) in arbitrary time \( t_f^k \) to an arbitrary value \( X_{1d} \) via a dead-beat controller [504] [294], chosen here as \( X_{1d} = (0, 0)^T \).

\[
\bar{v}_1(\tau) = B^T \exp(-A\tau)^T V^{-1}(t_f^k, t_0^k) V^T \left[ -\exp(-At_0^k) X_1(t_0^k) + X_{1d} \right] \quad (8.102)
\]

where \( V V^T = I_2 \), \( V^{-1}(t_f^k, t_0^k) V^T = \int_{t_0^k}^{t_f^k} \exp(-A\tau) BB^T \exp(-A\tau)^T d\tau \) is the controllability Grammian of the system \( \dot{X}_1 = AX_1 + B\bar{v}_1 \) with \( G(t, t_0^k) > 0 \). One therefore obtains

\[
X_1(t_f^k) = X_{1d} \quad (8.103)
\]

One has further to show that \( x_1(t) > 0 \) on \([t_0^k, t_f^k)\) and that \( \bar{v}_1 \) is calculable explicitly as a time-function for a suitable choice of \( A \). Otherwise one has to use some numerical approximation of the matrix exponential. Explicit calculation can be done when for instance \( a_1 = a_2 = 0 \) (24). In this case it can be computed that

\[
V^{-1}(t, t_0^k) V^T = \begin{pmatrix} \frac{(t-t_0^k)^3}{3} & \frac{(t_0^k)^2-t^2}{2} \\ \frac{(t_0^k)^2-t^2}{2} & t-t_0^k \end{pmatrix} \quad (8.104)
\]

Note that \( \det V^{-1}(t, t_0^k) V^T = \frac{(t-t_0^k)^4}{12} > 0 \) for all \( t > t_0^k \). Also recall that \( V^{-1}(t_f^k, t_0^k) V^T > V^{-1}(t, t_0^k) V^T \) for all \( t < t_f^k \), in the sense of positive definite matrices inequalities. Then the solution has the form

\[
X_1(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \left\{ \begin{pmatrix} 1 & -t_0^k \\ 0 & 1 \end{pmatrix} X_1(t_0^k) - G(t, t_f^k, t_0^k) \begin{pmatrix} 1 \\ -t_0^k \end{pmatrix} X_1(t_0^k) \right\} \quad (8.105)
\]

with

\[
G(t, t_f^k, t_0^k) =
\begin{pmatrix}
\frac{(t_0^k)^3}{3} + \frac{(t_f^k)^3-t^3}{4} & \frac{(t_f^k)^2-t^2}{6} - \frac{(t_0^k)^3-(t_f^k)^3}{6} \\
\frac{(t_0^k)^2-t^2}{2} + (t-t_0^k) \frac{(t_f^k)^2-(t_0^k)^2}{2} & \frac{(t_f^k)^2-t^2}{4} - \frac{(t_f^k)^3-(t_0^k)^3}{3}
\end{pmatrix}
\quad (8.106)
\]

Note that one should take care of the influence of the values of \( a_1 \) and \( a_2 \) on the obtained controller and above all on the obtained state trajectory: certainly one does not get the same behaviour when the matrix \( A \) has real negative eigenvalues, or complex conjugate eigenvalues. Such a fact might be important due to the unilateral constraint.
8.6. N-DEGREE-OF-FREEDOM RIGID MANIPULATORS

It is not obvious from (8.105) and (8.106) whether $x_1$ remains strictly positive on $[t^k_0, t^k_f)$ or not. If there is a shock before the stabilization on the surface $x_1 = 0$, then such a controller possesses no advantage over the one we have discussed above. Notice furthermore that such control strategy is basically open-loop (see (8.102)), and hence possesses all the drawbacks of such controllers. For instance if there is a disturbance or if the constraint position is not well-known, then such controller will not assure the stabilization on the surface, whereas the transition phase controllers derived above will be more robust to such uncertainties. Moreover in practice one may not be able to choose $t^k_f - t^k_0$ as small as desired due to input saturations in the motors (the size of the controller depends on $\lambda[I_k]$, see (8.102). As $t^k_f - t^k_0 \rightarrow 0$, one obtains an impulsive controller that shifts the state instantaneously to an arbitrary value). It may therefore happen that it has to be larger than with the controllers in (8.38) or (8.49). Control laws as in (8.102) have a theoretical interest only, and lack for practical feasibility. This is the reason why we have dealt throughout this chapter with controllers involving collisions phenomena, and that have been tested in practical experiments reported in the literature. It is however possible that deadbeat controllers might be useful in other applications of impacting systems.

In conclusion, unless $\mathcal{M}$ is a Jacobian, the constraints are orthogonal and the closed-loop decoupled system (8.78) is used, there is no proper decoupling at the collision instants $t_k$ between "normal" and "tangential" components. Then the stability of a complete task is more difficult to obtain. Therefore we conclude that orthogonality of the constraints is a very convenient assumption for existential problems, coherent meaning of the restitution rule, and for closed-loop stability analysis, i.e. for mathematical, mechanical and control purposes. This constitutes the main discrepancy between robotic tasks involving several regimes of motion and simple tasks involving only constrained motions, and is a consequence of rigid body generalized impacts models. An open problem is to find out a controller guaranteeing weak or strong stability, for possibly non-orthogonal constraints and a restitution rule of the form $\dot{q}(t^k_f) = \mathcal{E} \dot{q}(t^k_0)$, where $\dot{q}$ is a vector of velocities chosen as in chapter 6, subsection 6.5.4, paragraph General algorithm. The main difficulty (25) lies in the stability analysis of the impact Poincaré maps $P_{\Sigma,i}$, since a shock with $\Sigma_j$ may induce a jump in some components of $\tilde{x}_{\Sigma,i}$, $i \neq j$, and vice-versa.

Remark 8.29 Following remark 8.19 about the possible choices of a high-level switching control strategy, note that it is possible to define conditions $C_1$'s in (8.22) such that on $\Omega_{2k+1}$ (constrained phases), the controller switches to $u_t$ if there is an accidental detachment of the robot's tip from the constraint surface. This can be done by checking for instance whether $\lambda > 0$ or $\lambda = 0$ in (8.22), via a force sensor. It is clear that from a general point of view, robustness of the overall closed-loop scheme can be enhanced by both the low-level and high-level control algorithms. In particular it has been argued [342] that event-based switching conditions provide more robustness than open-loop ones: this is somewhat natural since this introduces

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25 Still, it is not clear at all whether this is more difficult than running 1500 m. in 3 min. and 40 s.. It might be argued that both things are one as useless as the other.
a kind of feedback through the switching strategy. In the $n$-degree-of-freedom case, the switching times can be also defined when the manipulator enters some region in the Cartesian space. Anyway, as long as existence of solutions is guaranteed and stability (weak or strong) can be proved, one can choose any control algorithm.

## 8.7 Another concept of stability

Motivated by some stability concepts that have been proposed in the related literature of hybrid dynamical systems [64](26), let us propose now the following lemma:

**Lemma 8.3 (rw-stability)** Let $x \in \mathbb{R}^l$, $y \in \mathbb{R}^p$, $z \in \mathbb{R}^q$ denote the system's closed-loop state vector on $\Omega_{2k}$, $\Omega_{2k+1}$ and $I_k$ respectively. Let $V_x(x)$, $V_y(y)$ and $V_z(z)$ be positive-definite functions satisfying $\alpha_i(||i||) \leq V_i(i) \leq \beta_i(||i||)$, $i = x, y$ or $z$, for some class $K$ functions $\alpha_i(\cdot)$ and $\beta_i(\cdot)$. Assume that

1. $V_x(T_{02k}^k) \leq V_x(T_{02k-2}^k)$ for all $k \geq 0$ and $V_x(x) \leq -\gamma_x(||x||)$, $\sigma_{V_x}(t_k) \leq 0$ on $\Omega_{2k}$, $k \geq 0$, $V_x$ uniformly bounded on $\cup_k \Omega_{2k}$.
2. $V_y(T_{02k-1}^{2k-1}) \leq V_y(T_{02k-3}^{2k-3})$ for all $k \geq 0$ and $V_y(y) \leq -\gamma_y(||y||)$ on $\Omega_{2k+1}$, $k \geq 0$, $V_y$ uniformly bounded on $\cup_k \Omega_{2k+1}$.
3. $\sigma_{V_z}(t_k) \leq 0$, $V_z(t_{k-1}^-) \leq V_z(t_k^+)$, $V_z$ is uniformly bounded on $I_k$, $k \geq 0$, and $\{t_k\}$ possesses a finite accumulation point.

If a) and b) are satisfied and $\lambda[\cup_k \Omega_{2k}] = +\infty$, $\lambda[\cup_k \Omega_{2k+1}] = +\infty$, then $x \to 0$ as $t \to +\infty$, $t \in \cup_k \Omega_{2k}$, and $y \to 0$ as $t \to +\infty$, $t \in \cup_k \Omega_{2k+1}$. Moreover if c) is satisfied, we can require similar conditions as in lemma 8.2 so that the mappings $P_{\Sigma,i} : \bar{z}_{\Sigma,i}(t_k^+) \mapsto \bar{z}_{\Sigma,i}(t_{k+1}^+)$ are Lyapunov stable, where $\bar{z}_{\Sigma,i}$ is defined similarly as $\bar{z}_{\Sigma,i}$ in lemma 8.2.

The proof follows from the fact that each positive definite function $V_i$, $i = x, y$, has a derivative that satisfies the required criterion for asymptotic stability along the trajectories of the closed-loop system. Due to the additional conditions on decrease of $V_i$ between the respective phases, the convergence of the state vectors $x$ and $y$ on their respective phases follows since the phases are assumed to have infinite Lebesgue measure.

**Remark 8.30** It is meaningless to suppose that a unique system possesses several state vectors, since a dynamical system is defined with one state vector. What is meant in lemma 8.3 is that contrarily to the concepts of lemmas 8.1 and 8.2, we analyze the system as if it was the time-concatenation of several subsystems.

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Note however that contrarily to what follows, a system of switching continuous vector fields with a common state vector is considered in [64]. Lyapunov stability is shown.
8.7. ANOTHER CONCEPT OF STABILITY

indexed by $x$, $y$ and $z$. When the corresponding phase is activated, we look at the variation of the corresponding state vector and disregard it during the rest of the time, just assuming that it remains bounded. Hence $x$, $y$ and $z$ are to be considered as some time-functions whose variation is observed only on certain windows. The following conditions will guarantee that they converge to zero during the periods of interest. Actually the system's state $x_0$ is made of a certain combination of $x$, $y$ and $z$: if they are independent, then $x_0^T = (x^T, y^T, z^T)$. In practice $x$, $y$ and $z$ may differ mainly because of different dynamical feedbacks on the respective phases. Such manipulation is exactly the same thing as considering that a permanently constrained mechanical system has reduced order, or that it has full order but with some state variables set to a constant value. For instance using the MacClamroch-Wang's variables, one may consider that the system's state is $x_2$ (lemma 8.3 concept), or that it is $(x_1, x_2)$, with $x_1 = 0$ (lemmas 8.1 and 8.2 concepts). Note that this imposes conditions on the vector field and on the initial conditions. But in the framework of the systems we study, the transition from phases $\Omega_{2k}$ to phases $\Omega_{2k+1}$ is assured via $I_k$. This allows us to avoid problems related to initial conditions in $x_2$ and a description via descriptor variable systems on $\Omega_{2k+1}$: indeed $\Omega_{2k+1}$ are always initialized with $x_1(T_{02k+1}) = 0$. Finally it is clear that we could have considered an intermediate stability concept, using several positive-definite functions and a unique state vector $x$ as for weak stability.

We denote this stability concept rw-stability since it corresponds to a relaxed form of weak-stability introduced in lemma 8.1. Clearly w-stability $\Rightarrow$ rw-stability, but not the contrary in general. Contrarily to the stability concept introduced in lemmas 8.1 and 8.2, we do not require here the use of a unique positive-definite function $V$, but we allow for a particular $V_i$, $i = x, y, z$, for each different type of phases. Hence we consider that the closed-loop state may be different from one phase to another, and we study its variation only during the corresponding phases. Evidently there must be a link between the values taken by $V_i$ from one phase to the next one, which is apparent in the lemma conditions.

One question is: is it advantageous to use rw-stability instead of w-stability? In other words, can we do more than previously in terms of stabilization with various controllers? Obviously, the ultimate goal we pursue is to be able to find conditions such that, given $u_{nc} \in U$ stabilizing free-motion tasks, $u_c \in U$ stabilizing constrained-motion tasks, and $u_t \in U$ stabilizing the transition phase, we can combine (i.e. determine switching times or events) these controllers so that one stability criterion or another is satisfied. Does rw-stability allow for more freedom in the choice of those switching times? Most importantly, does it allow us to avoid the problem of non-decoupling between "tangential" and "normal" components raised in subsection 8.6.2 on control of $n$-degree-of-freedom manipulators with non-integrable transformed velocities?

Recall also that we want to obtain w- or rw-stability with arbitrary durations of the phases $\Omega_{2k}$ and $\Omega_{2k+1}$. It is clear that from a general point of view, if the closed-loop state vectors $x$, $y$ and $z$ are independent, then rw-stability is a useful tool. This
is not the case here since \( q \) and \( \dot{q} \) are always part of the state. Let us illustrate the use of lemma 8.3 for the system in (8.91). In the following \( X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \), see remark 8.6.

- On \( \Omega_{2k} \)

\[
\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} u_{nc} = \bar{M}(X)(\dot{X}_d - \lambda \dot{X}) + \bar{C}(X, \dot{X})(\dot{X}_d - \lambda \dot{X}) + \bar{g}(X) - \Lambda S \quad (8.107)
\]

where \( H(q, \dot{q}) = \bar{C}(X, \dot{X}) \dot{X} + \bar{g}(X) \) and the matrix \( \frac{d}{dt}(\bar{M}(X)) - 2\bar{C}(X, \dot{X}) \) is skew-symmetric \([91]\), \( S = \dot{X} + \lambda \ddot{X} \), \( \Lambda = \Lambda^T > 0 \). The closed-loop system on \( \Omega_{2k} \) is therefore given by

\[
\begin{aligned}
\bar{M} \dot{S} + \bar{C} S + \Lambda S &= 0 \\
\dot{X} &= -\lambda \ddot{X} + S
\end{aligned}
\quad (8.108)
\]

Hence we can consider \( x = (\dot{X}, S) \in \mathbb{R}^{2n} \) and stability can be shown with the function \([509]\)

\[
V_x(x) = \frac{1}{2} S^T \bar{M} S + \lambda \dot{X}^T \Lambda \dot{X}
\quad (8.109)
\]

whose derivative along trajectories of (8.108) satisfies

\[
\dot{V}_x(x) \leq -\gamma_x(||\dot{X}, S||) \leq 0
\quad (8.110)
\]

- On \( \Omega_{2k+1} \)

\[
\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} u_c = \begin{pmatrix} \bar{M}_{12}(\ddot{x}_{2d} - \lambda_1 \dot{x}_2 - \lambda_2 \ddot{x}_2) + H_1 - p_{z,d} \\ \bar{M}_{22}(\ddot{x}_{2d} - \lambda_1 \dot{x}_2 - \lambda_2 \ddot{x}_2) + H_2 \end{pmatrix}
\quad (8.111)
\]

The closed-loop dynamics are given by \( x_1 \equiv 0, \dot{x}_2 - \lambda_1 \dot{x}_2 - \lambda_2 \ddot{x}_2 = 0, p_{x,1} = p_{z,d} \geq 0, V_y(y) = y^T P y, y^T = (x_2^T, \dot{x}_2^T) \in \mathbb{R}^{2(n-m)} \).

- On \( I_k \)

We apply the same controller as in subsection 8.6.2 to obtain \( t_\infty < +\infty \) and \( x_1(t_\infty) = \dot{x}_1(t_\infty) = 0 \).

The crucial point is to verify if \( V_x(T_f^{2k}) \leq V_x(T_f^{2k-2}) \) and if \( V_y(T_f^{2k-1}) \leq V_y(T_f^{2k-3}) \). The jumps in \( V_x \) at eventual collisions are generically computed as

\[
\sigma_{V_x}(t_k) = \frac{1}{2} S^T(t_k^+) \bar{M} S(t_k^+) - \frac{1}{2} S^T(t_k^-) \bar{M} S(t_k^-)
\quad (8.112)
\]
8.7. ANOTHER CONCEPT OF STABILITY

Using that \( S(t_k^+) = \dot{X}(t_k^+) + \lambda X(t_k) - \dot{X}_d(t_k) - \lambda X_d(t_k) \) and assuming \( X_d \equiv 0 \) at the discontinuity-times, one gets

\[
\sigma_V E(t_k) = T_L(t_k) + \lambda \left( 0, x_2^T(t_k) \right) \tilde{M}(t_k) \dot{X}(t_k^-) \tag{8.113}
\]

The kinetic energy loss \( T_L(t_k) \) is negative. But in general it may be that \( \sigma_V E(t_k) > 0 \), and similarly \( \sigma_V F(t_k) \) may be positive. The same conditions on \( \lambda_1, \lambda_2 \) and \( a \) as in subsection 8.6.2 (after equation (8.99)) can be supposed to hold to get \( V_F(T_k^{2k-1}) \leq V_F(T_f^{2k-3}) \), by insuring that \( V_F(t_k^+) - V_F(t_k^-) \geq \sigma_V E(t_k) \). But we also have to derive a similar condition for \( V_Z \), so that the task can be rw-stabilized with arbitrary \( \lambda[\Omega_{2k}] \) and \( \lambda[\Omega_{2k+1}] \). We have

\[
V_Z(t_k^+) - V_Z(t_k^-) = \int_{t_k^+}^{t_k^+} \left( \tilde{M} \tilde{S} + \tilde{S}^T \frac{\tilde{M}}{2} S + 2\lambda \tilde{X}^T \lambda \tilde{X} \right) dt \tag{8.114}
\]

If \( X_d \equiv 0, \dot{S} = \dot{X} + \lambda \tilde{X} = \left( \begin{array}{c} -a \\ v_2 \end{array} \right) + \lambda \tilde{X} \) and \( \tilde{S}^T \tilde{M} \tilde{S} = (\tilde{X} + \lambda X)^T \tilde{M} \left( \begin{array}{c} -a + \lambda \tilde{x}_1 \\ v_2 + \lambda \tilde{x}_2 \end{array} \right) \).

Conditions guaranteeing that \( V_Z(t_k^+) - V_Z(t_k^-) \leq 0 \) are not evident.

Now suppose that instead of applying the Slotine and Li controller \([501]\) in (8.107), we rather apply the scheme of Paden and Panja \([419]\) on \( \Omega_{2k} \). We obtain

\[
\left[ \begin{array}{c} T_1 \\ T_2 \end{array} \right] u_{nc} = \tilde{M} \dot{X}_d + \bar{C} \dot{X}_d + \bar{g} - \gamma_1 \dot{X} - \gamma_2 \dot{X} \tag{8.115}
\]

with

\[
V_Z(x) = \frac{1}{2} x^T \tilde{M} \tilde{X} + \frac{1}{2} \gamma_1 \dot{X}^T \dot{X} \tag{8.116}
\]

with \( x^T = (\tilde{X}^T, \dot{\tilde{X}}^T) \in \mathbb{R}^{2n} \). Then if \( X_d \equiv 0 \) on \( I_k \) we obtain

\[
\sigma_V E(t_k) = T_L(t_k) \leq 0 \tag{8.117}
\]

and

\[
T_{t_k^{+}} = e^{2T(t_k^{-})} \tag{8.118}
\]

Now if the Paden and Panja scheme is applied on \( I_k \) with \( X_d \) constant, we obtain

\[
V_Z(t_k^+) - V_Z(t_k^-) = - \int_{t_k}^{t_{k+1}} \gamma_1 \dot{X}^T \dot{X} dt + \sigma_T(t_{k+1}) \leq 0 \tag{8.119}
\]

Hence the restriction to the section \( \Sigma_2 = \{ x_{1,i} = 0 \} \) of \( V_Z(x) \) in (8.116) and with \( X_d \) constant qualifies as a Lyapunov function \( V_{\Sigma_2,i} \Delta \tilde{V}_{\Sigma_2,i} \) for the mapping \( P_{\Sigma,i} \) (see lemma 8.3). Note that here from the simple form of the constraint \( (x_1 = 0) \) we can choose \( \tilde{z}_i = z \) (the mapping \( G_z \) which associates to the "flight-times" state vector \( z \) the transformed state vector \( \tilde{z}_i \) which is more suitable for the study of the impact Poincaré map, is the identity). The closed-loop system is then given by:

\[
\tilde{M} \ddot{X} + \bar{C} \dot{X} + \gamma_1 \dot{X} + \gamma_2 \dot{X} = 0 \tag{8.120}
\]
CHAPTER 8. FEEDBACK CONTROL

From the fact that for any initial condition for \( X \) and \( \dot{X} \), \( X \to X_d \), it suffices that \( x_{id} < 0 \) to insure that \( \{t_k\} \) exists and is infinite (moreover the state \( z(t) \) is bounded, see (8.119)). Then from (8.118) and (8.119) it follows that \( V(t_k) \to 0 \) exponentially as \( k \to +\infty \). However it remains to be proved that such a controller \( u_t \) guarantees finite-time stabilization on \( x_1 = 0 \).

Note that if the same controller \( u_t \) as in subsection 8.6.2 is applied we obtain

\[
V_s(t_{k+1}^-) - V_s(t_k^+) = \int_{t_k}^{t_{k+1}} \left[ X^T \dot{M} \left( \begin{array}{c} -a \\ v_2 \end{array} \right) + \frac{\dot{X}^T \ddot{M}}{2} \dot{X} + \gamma_1 X^T X \right]
\]

whose sign is not assured.

In conclusion, the Paden and Panja scheme (which guarantees tracking during free-motion phases) is more suitable than the Slotine and Li scheme to guarantee rw-stability. If it was possible to guarantee finite-time stabilization on \( x_1 \equiv 0 \) with the dynamics in (8.120), then the rw-stability concept would allow us to design a control strategy using two different controllers (the Paden and Panja scheme on \( \Omega_{2k} \) and \( I_k \) and a decoupling-linearizing scheme on \( \Omega_{2k+1} \)), yielding two different closed-loop state vectors \( x \in \mathbb{R}^{2n} \) and \( y \in \mathbb{R}^{2(n-m)} \). This is not possible with the weak-stability concept of lemma 8.1. It remains however to decide whether this is a significant improvement for practical purposes of control of manipulators subject to unilateral constraints.

Remark 8.31 Let us consider the (partial) open-loop dynamics in (8.78). Let us define the signal:

\[
s = M \ddot{q} - M \ddot{q}_d + M \Lambda (q - q_d)
\]

\[
= \ddot{q} - M \ddot{q}_d + M \Lambda \ddot{q}
\]

Let us choose the input

\[
v = M \ddot{q}_d + M \ddot{q}_d - M \Lambda \ddot{q} - M \Lambda \ddot{q} - M \dot{q} + M \dot{q}_d - M \Lambda \ddot{q}
\]

Then we obtain the closed-loop system:

\[
\dot{s} + s = 0
\]

from which it follows that \( s \to 0 \) exponentially fast. From the fact that \( s = M(\ddot{q} + \Lambda \ddot{q}) \) and since \( M \in \mathbb{R}^{n \times n} \) is full-rank, one concludes that \( \ddot{q} \to 0 \) asymptotically as well. The free-motion positive definite function \( V \) can be chosen as \( V = s^T s \). It then remains to check all the conditions that are required to get weak or strong stability, and this may not be easier than with the foregoing controllers. Therefore although it is possible to use the dynamics in (8.78) and obtain tracking results on \( \Omega_{2k} \) without taking care of the integrability of the transformed velocities \( \ddot{q} = M \ddot{q} \), it is not guaranteed that this is of any help for the overall robotic task stabilization. More precisely, recall that the first component of \( M \ddot{q} \) is \( n^T q \dot{M}(q) \ddot{q} = \nabla_q f(q) \ddot{q} = \dot{q}_\text{norm} \). Hence if the transition phase controller \( u_t \) guarantees that \( \dot{q}_\text{norm}(t_f) = 0 \),
and if $q_d = 0$, the first component of $s$, say $s_{\text{norm}}$, satisfies $s_{\text{norm}}(t_f^f) = M\Lambda q(t_f^f)$. Now if $\Lambda = \lambda I_n$, one gets $s_{\text{norm}}(t_f^f) = \lambda \nabla_q f(q)^T q(t_f^f)$. It is clear that in general, $q_{\text{norm}} = f(q) = 0 \neq \nabla_q f(q)^T q = 0$. But if one uses the quasi-coordinate $q_1 = f(q)$, then $\nabla_q f(q) = (1, 0, \ldots, 0)^T$ so that $\nabla_q f(q)^T q = q_1$. Therefore in this case one gets $s_{\text{norm}}(t_f^f) = 0$ if $q_{\text{norm}}(t_f^f) = q_{\text{norm}}(t_f^f) = q_d = 0$. This could be used to design a weakly stable closed-loop scheme when the transformed velocities are not integrable, and should be the object of future studies.

Another path may be to choose another function $V$. For instance the choice $V = \sum_{i=1}^n s_i$ may be done as suggested in [409], and using the theory about non-smooth Lyapunov functions [489]. We do not investigate further the use of non-smooth Lyapunov functions which would require to recall basic theoretical facts in close relationship with Clarke's generalized gradient (see appendix D) (27). Let us reiterate that the debate is far from being closed, and that it might be that the right Lyapunov function has not yet been discovered.

8.8 Examples

In this section the foregoing developments are illustrated on several simple cases. As it is expected, the transformed velocity $\dot{q}$ is not integrable in general.

Example 8.1 A two-joint prismatic manipulator as in figure 8.7 has inertia matrix

$$M(q) = \begin{pmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{pmatrix}.$$  

The constraint is a vertical straight line, i.e. $f(q) = d - q_2 \geq 0$. Then $q_{\text{norm}} = -q_2$ and $q_{\text{ang}} = q_1$.

---

(27) Let us note that if such an analysis is used, we end up with non-smooth dynamical systems represented by a class of measure differential equations, and non-smooth (in time and state variables) Lyapunov functions. Nonsmoothness is everywhere in this control problem!
Example 8.2 The manipulator depicted in figure 8.8 has inertia matrix
\[ M(q) = \begin{pmatrix}
  m_1 + m_2 & \frac{1}{2}m_2l_2G_2 \\
  \frac{1}{2}m_2l_2G_1 & I_2 + m_2l_2^2
\end{pmatrix}. \]
The constraint is given by \( f(q) = -l_2\cos(q_2) + d \geq 0 \). Then \( q_{\text{norm}} = -l_2\cos(q_2) + d \), and \( q_{\text{tang}} = (m_1 + m_2)q_1 + \frac{1}{2}m_2l_2G_2q_2 \).

Example 8.3 Consider now a 2-degree-of-freedom planar manipulator with two revolute joints as in figure 8.9. Then the inertia matrix takes the form
\[ M(q) = \begin{pmatrix}
  a_1 + 2a_3\cos(q_2) & a_2 + a_3\cos(q_2) \\
  a_2 + a_3\cos(q_2) & a_2
\end{pmatrix}, \]
with \( a_1 = m_1l_2^2G_1 + I_1 + I_2 + m_2(l_1^2 + l_2^2) \), \( a_2 = m_2l_1l_2G_2, \) and \( a_3 = m_2l_1l_2G_1 + m_2l_2^2G_2 + I_2 \). The constraint is given by \( f(q) = -l_1\cos(q_1) - l_2\cos(q_1 + q_2) + d \geq 0 \). Then \( q_{\text{norm}} = -l_1\cos(q_1) - l_2\cos(q_1 + q_2) \). One gets \( \dot{q}_{\text{tang}} = g_1(q_1, q_2)\dot{q}_1 + g_2(q_1, q_2)\dot{q}_2 \). A necessary condition for \( \dot{q}_{\text{tang}} \) to be integrable (i.e. there exists \( q_t \) such that \( \dot{q}_t = \dot{q}_{\text{tang}} \)) is that \( \frac{\partial^2 q_1}{\partial q_2 \partial q_1} = \frac{\partial^2 q_2}{\partial q_1 \partial q_2} \). This is not the case. Hence this simple case is not integrable. One may search for the existence of constraints such that it is.

The last example proves the importance of studying closed-loop schemes for the non-integrable case.
8.9 Conclusions

In this chapter we have studied the control of rigid manipulators subject to unilateral constraints on the position. We have reviewed the basic facts on the dynamics of such systems, which are complex hybrid dynamical systems merging ordinary and measure differential equations, algebraic constraints and possibly some high-level control strategy. In particular, attention has been focussed on the definition and existence of the solutions of the closed-loop system as functions of local bounded variation in time, and on the definition of the restitution rules. We have proposed a stability analysis framework that mixes and adapts some basic stability analysis for measure differential equations, hybrid dynamical systems and discrete-time Poincaré impact maps. In fact this stability concept is based on the separation of phases where the smooth vector field fixed point belongs to Int(\(\Phi\)), and those where this fixed point is outside \(\Phi\) or on \(\partial\Phi\). In the first case we use stability concepts for measure differential equations as developed in [27], whereas in the second case we study the Lyapunov stability of impact Poincaré maps. The proposed closed-loop stability concept is natural since it is close in spirit to Lyapunov’s second method and merges stability of continuous and discrete time systems, which are at the core of the dynamics of mechanical systems submitted to unilateral constraints. This allowed us to analyze various discontinuous control strategies, and is expected to provide a convenient stability analysis framework for subsequent extension of the results towards other controllers (such as adaptive controllers) and for robustness analysis (such as bad-timing effects for the switching times between different controllers, or bad knowledge of the constraint position). We have focussed first on a one-degree-of-freedom example, and we have introduced various controllers to illustrate the different types of stability criteria we have proposed. Then we have discussed the \(n\)-degree-of-freedom case, and we have highlighted some difficulties related to the possible non-decoupling between the "normal" and the "tangential" parts of the velocities during the rebounds. We hope that the developed tools provide a first satisfactory theoretical answer to Professor Paul’s statement [422] that the contact problem is unsolved for rigid manipulator, rigid sensor, rigid environment problems., that is naturally embedded into the class of systems with unilateral constraints. It is clear that experimental studies for the validation of an impact model for \(n\)-degree-of-freedom manipulators striking a rigid obstacle, and for the validation of the proposed control strategies, are a necessary future step.

Remark 8.32 Strictly speaking, bipedal robots do not belong to the class of systems studied in this chapter. Indeed the free-motion system (no foot in contact with the ground) is not controllable (the system’s center of gravity motion depends only of initial conditions at the beginning of the flight-phase, according to the very basics of classical mechanics, since control torques at the joints are internal actions). Moreover the goal of feedback control is different (see for instance Hurmuzlu’s studies [221] [98]), hence the stability analysis framework. In particular, the phases \(I_k\) may reduce to one impact time \(t_0\) (but this is not mandatory: even if there is only one impact, one may have \(\lambda[I_k > 0]\) if plastic impacts are assumed like in
CHAPTER 8. FEEDBACK CONTROL

[221]. Also the system always "slides" on one constraint \(^{(28)}\) surface when walking is considered (there is always a foot in contact with the ground). In other words, if we denote the two constraints as \(f_1(q) \geq 0\) and \(f_2(q) \geq 0\), the walking motion is such that \(f_i(q(t)) = 0\) for some \(i \in \{1, 2\}\) and all \(t \geq \tau_0\). Thus there are no free-motion phases in the sense we have defined in this chapter. In this sense the dynamics of a bipedal robot are quite similar to those of the rocking block (except of course that the dynamics of bipedal robots between impacts are highly nonlinear). In fact one wants to produce \(via\) feedback control a shock (or a series of shocks) alternatively with each one of the constraints (i.e. \(t_k = t_{k}^{1}, t_{k+1} = t_{k+1}^{2}\)), while controlling the motion between impacts, the whole motion being periodic. Obtaining a good model and a believable behaviour at the impacts is already a difficult task (see the developments in chapter 6, section 6.5). But even after some simplifying assumptions have been done \(^{(29)}\) (no slipping at contact, plastic impacts -the tip of the swing limb does not rebound but keeps contact after the shock-, controller of sufficiently large dimension), the control problem remains difficult. Hurmuzlu [221] deals with a 5 links planar bipede, with input torques at each joint and at the ankles. He first fixes some relationships between the system's coordinates, and analyzes the influence of various parameters on the percussion vector (i.e. is there a percussion at both contacts or only at the foot which strikes the ground) when these relationships are verified. Then he chooses a feedback linearizing-decoupling controller and studies numerically the closed-loop behaviour \(via\) an impact Poincaré map. However this differs from the concept of lemma 8.2, because the Poincaré map is defined from one impact with one foot to the next impact with the other foot. Hence within the framework we developed, this is just like looking at the system at discrete instants, from \(I_k\) to \(I_{k+1}\) \(^{(30)}\). In fact Hurmuzlu chooses some closed-loop dynamics between the impacts and studies the effect of those dynamics on the chosen impact Poincaré map: is there a periodic trajectory? Is it stable? Chang and Hurmuzlu [98] propose a sliding-mode feedback controller and use similar tools to analyze closed-loop dynamics. A simple model of hopping robot is analyzed in [159], where it is assumed that the shock is purely inelastic and that there is no sliding during the contact phase. Both impulsive and bounded controllers are derived to stabilize the system around a prescribed periodic trajectory. It is clear that there is a strong similarity between the juggling robots and such hopping robot. Indeed both systems possess a codimension one unilateral constraint, and both have non-controllable free-motion dynamics. It presently remains an open question to know

\(^{(28)}\)Sliding is in the configuration space. The foot may quite be fixed with respect to the ground.

\(^{(29)}\)Which correspond to choosing a particular generalized restitution rule when a singularity is attained, see chapter 6.

\(^{(30)}\)One sees that the Poincaré map hence defined decreases the dimension of the total system by 2, since the Poincaré section has codimension 2 (the impacts occur always at the singularity of the domain \(\Phi\). In fact in [221], the author chooses to eliminate one component of the generalized coordinate vector, the one that corresponds to \(f_i(q) = 0\), and works with a reduced-dimension system. Hence the Poincaré section looks like a codimension 1 subspace. In other words, one has the choice between working with a \((2 + n)\)-dimensional system subject to a codimension 2 constraint, or working with an \(n\)-dimensional system, but with additional switching conditions in the model that incorporate the change of supporting foot.
whether it is sufficient to classify mechanical systems with unilateral constraints into two main classes (controllable and non-controllable free-motion dynamics) to be able to derive general feedback control methods, or if it is necessary to refine this classification, as discussed in chapter 1.

We do not develop further neither the dynamics nor the control of bipedal robots. It is however note worthy that such systems belong to the family of (controlled) multibody-multiconstraints mechanical systems which are the topic of this monograph. An open problem may be to derive a very general stability concept which would apply to any system in the family. We believe that the introduced concepts of weak and strong stability are one step towards this aim, since anyway a stability concept for this class of systems must possess an hybrid-dynamical systems and multiple-collision coloration (whether the ultimate goal is tracking of some reference trajectory of the state or part of it, or stabilization of limit cycles).

**Remark 8.33** In accordance with the discussion at the end of chapter 7, let us note that if one is able to prove that the solutions of a closed-loop compliant approximating problem $P_n$ converge to those of the rigid case $x$ (in other words, by replacing the constraints by a spring+damper-like model $k_n$, $f_n$, one is able to show that the closed-loop solution $x_n$ converges to $x$), then for large enough $n$, $x_n$ has the same behaviour as $x$. This could be used in particular to show some robustness properties of the controllers designed from a rigid body approach, when some small enough flexibilities are present at the contact point.

**Remark 8.34** In relationship with the impact Poincaré maps that have been defined throughout chapters 7 and 8, let us recall that strictly speaking, the map defined in lemma 8.2 should involve the impact times $t_k$, as long as the considered closed-loop system is not autonomous. Indeed in this case the flight-times $\Delta_k$ do not depend only on the post-impact state values. This is the reason why the bouncing-ball map is two-dimensional, since the table excitation is an exogeneous signal. However we have preferred to call the defined mappings Poincaré maps for simplicity, and also to be because we needed to use the same state vector for both the transition phase and the continuous phases. Moreover the conditions stated in lemma 8.2 include that the sequence $\{t_k\}$ does possess a finite accumulation point (a fact that the complete Poincaré map stability should indeed include, see e.g. [574]).

**Remark 8.35** Finally let us note that the overall stability analysis of the whole robotic task, taking into account the low-level as well as the high-level controllers and state-spaces, is a hard work involving future fundamental studies on hybrid dynamical systems. Roughly, Lyapunov stability in normed spaces for continuous-time systems is quite an established theory [554], and its counter part for discrete-event systems has recently received attention in the fundamental paper [421]. One of the main difference between both is that in the second case, the state-space cannot even be supposed to be a vector space, and one has to be content with metric spaces. The interaction between differential equations and finite automata, with emphasis
on the continuity properties of maps from the "continuous" state space $\mathcal{C}$ to the
"discrete-event" state-space $\mathcal{D}_e$ (that involves the research of topologies of $\mathcal{D}_e$ different from the discrete topology $\mathcal{D}_e$) has been studied in [62]. It is outside the scope of this monograph to develop further this topic. We simply mention that control of mechanical systems subject to unilateral constraints in the configuration space involves problems of mathematical nature (measure differential equations, state-spaces of $RCLBV$ functions), mechanical nature (models of impact), dynamical-systems-theory nature (stability of solutions), and is naturally embedded in the class of hybrid dynamical systems.

\footnote{Here we speak about topology in the mathematical sense, that allows one to define in a given set collections of open subsets, i.e. subsets verifying certain basic properties, see e.g. [480].}
Appendix A

Schwartz’ distributions

There are mainly two classes of readers: those who possess a strong enough mathematical background and who do not need to be explained what is a distribution, a measure, a function of bounded variation ..., and those who are really not confident with all those analytical tools (among them, the author of this work!). Most of the appendix is for the second class. It is clear that such (rather complex) mathematical tools cannot be swallowed just by reading some definitions and theorems at the end of a book devoted to impact dynamics. Since however nonsmooth dynamics cannot be understood without even very basic knowledge on measures and distributions theories, it is preferable to provide some notions which, perhaps, will be a motivation to go further. Basic facts about dissipativity of controlled systems are also given, but it is assumed that the reader is confident with Lyapunov stability theory. A complete exposition of Lyapunov stability theory can be found for instance in Vidyasagar’s book [554].

We recall in this appendix some basic facts about two definitions of a distribution, the first one more related to rigid bodies percussions, the second one related to convergence of compliant problems towards rigid problems. We refer to the classical books [478] and [12] for more details.

A.1 The functional approach

In this section we first briefly introduce the functional notion of a distribution as defined in [478].

**Definition A.1** $\mathcal{D}$ is the subspace of smooth $^1$ functions $\varphi : \mathbb{R}^n \to \mathbb{C}$, with bounded support.

Thus a function $\varphi$ on $\mathbb{R}^n$ belongs to $\mathcal{D}$, if and only if $\varphi$ is smooth, and there exists a bounded set $K_{\varphi}$ of $\mathbb{R}^n$ outside of which $\varphi \equiv 0$. As an example, L. Schwartz

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$^1$i.e. indefinitely differentiable.
APPENDIX A. SCHWARTZ' DISTRIBUTIONS

gives the following function [478] chapter 1 §2, with \( n = 1, K_\varphi = [-1, 1] \):

\[
\varphi(t) = \begin{cases} 
0 & \text{if } |t| \geq 1 \\
 e^{i\pi/n} & \text{if } |t| < 1
\end{cases}
\] (A.1)

**Definition A.2** A distribution \( D \) is a continuous linear form defined on the vector space \( D \).

This means that to any \( \varphi \in D \), \( D \) associates a complex number \( D(\varphi) \), noted \( \langle D, \varphi \rangle \). The space of distributions on \( D \) is the dual space of \( D \) and is noted \( D^\ast \).

The functions in \( \mathcal{D} \) are sometimes called test-functions.

Two distributions \( D_1, D_2 \) are equal on an open interval \( \Delta \) if \( D_1 - D_2 = 0 \) on \( \Delta \), i.e. if for any \( \varphi \in D \) whose support \( K_\varphi \) is contained in \( \Delta \), then \( \langle D_1 - D_2, \varphi \rangle = 0 \). In fact, one can generate a distribution from any locally integrable function \( f \), via the integral \( \int_{K_\varphi} f(x)\varphi(x)dx \). However some distributions cannot be generated by locally integrable functions, like for instance the Dirac distribution and its derivatives. They are called singular distributions (or sometimes generalized functions).

Contrarily to functions, all distributions (i.e. elements of \( D^\ast \)) are infinitely differentiable. The \( m \)th derivative of \( T \in D^\ast \) is given by \( \langle T^{(m)}, \varphi \rangle = (-1)^m \langle T, \varphi^{(m)} \rangle \), for all \( m \in \mathbb{N} \).

An important feature of distributions is that in general, the product of two distributions does not define a distribution. A simple example that we encounter in dynamics of systems with unilateral constraints when we deal with the work of the impulsive forces is that of a distribution generated by a function \( f(t) \) discontinuous at \( t = 0 \) and the Dirac distribution \( \delta_0 \): the product makes no sense since \( \langle f(t)\delta_0, \varphi(t) \rangle = \langle \delta_0, f(t)\varphi(t) \rangle \) that should be equal for any \( \varphi \in D \) to \( f(0)\varphi(0) \): but \( f(0) \) is not defined (only the right and left limits are).

**Remark A.1** One may wonder why L. Schwartz has chosen the space \( D \) to define distributions. In fact other spaces of functions can be chosen. But note that basically, a distribution \( T \) is given by an integral \( \int T(x)\varphi(x)dx \). Hence the more "regular" the function \( \varphi \) is, the more "irregular" \( T \) may be to assure that the product \( T(x)\varphi(x) \) is integrable. In other words, if one considers two spaces of functions \( D_1 \) and \( D_2 \) with \( D_1 \subset D_2 \) with \( D_1 \subset D_2 \) satisfy \( D_2 \subset D_1 \) (2). This (very) roughly explains why the space \( D^\ast \) is a very "big" space, because the requirements imposed on the elements of \( D \) are very stringent (what else can one ask to a function than being infinitely differentiable on a compact interval?).

For instance consider \( D_1 \) to be the space of continuous functions with bounded support. Then \( \langle \delta_0, \varphi_1 \rangle = - \langle \delta_0, \varphi_1 \rangle \), but this expression is not defined for all \( \varphi_1 \in D_1 \), because \( \varphi_1 \) may be discontinuous at 0. Hence \( \delta_0 \notin D_1^\ast \).

\(^2\)Actually, this is not so simple. Some tighter relationships between \( D_1 \) and \( D_2 \) have to be verified.
Remark A.2 The space of functions $\mathcal{D}$ is sometimes denoted as $K$ [182]. Another classical space of functions whose dual is used in the theory of distributions, is the space of smooth functions that decrease faster than $\frac{1}{|x|^\alpha}$ for any $\alpha \in \mathbb{R}$, and whose derivatives of all orders satisfy the same constraint.

A.2 The sequential approach

Schwartz' distributions can also be defined via the sequential approach [12] §4.3, in relation with [478] theorem 23. Roughly, one starts by defining fundamental sequences of continuous functions on a fixed interval $(a, b)$, and then a relation of equivalence between fundamental sequences. This way of introducing distributions is similar to introducing real numbers as limits of sequences of rational numbers (recall that any real number can be approximated by a sequence of rational numbers, i.e. the set of rational numbers is dense in $\mathbb{R}$). Distributions can thus be defined as limits of sequences of continuous functions: but there are sequences of continuous functions that do not converge towards a function, as it is well-known e.g. for the Dirac distribution: this shows that the space of functions has to be completed by other mathematical objects, that one calls distributions.

Definition A.3 A sequence $\{f_n(\cdot)\}$ of continuous functions defined on $(a, b)$ is fundamental if there exist a sequence of functions $\{F_n(\cdot)\}$ and an integer $k \in \mathbb{N}$ such that

- $F_n(\cdot) = f_n(\cdot)$ for all $x \in (a, b)$.
- $\{F_n(\cdot)\}$ converges almost uniformly

A sequence of functions converges almost uniformly (a.u.) on $(a, b)$ if it converges uniformly on any interval $[c, d] \subset (a, b)$ (for instance, $\{\frac{x}{n}\}$ converges a.u. towards 0 on $(-\infty, +\infty)$. Before defining distributions, one needs to define equivalent fundamental sequences:

Definition A.4 Two fundamental sequences $\{f_n(\cdot)\}, \{g_n(\cdot)\}$ are equivalent if there exist $\{F_n(\cdot)\}, \{G_n(\cdot)\}$ and $k \in \mathbb{N}$ such that

- $F_n(x) = f_n(x)$ and $G_n(x) = g_n(x)$.
- $\{F_n(\cdot)\}$ and $\{G_n(\cdot)\}$ converge a.u. towards the same limit.

One denotes $\{f_n(\cdot)\} \sim \{g_n(\cdot)\}$.

It can be proved that $\sim$ is a relation of equivalence. Hence it allows one to construct equivalence classes of fundamental sequences of continuous functions. The equivalent class to which $\{f_n(\cdot)\}$ belongs is denoted as $[f_n(\cdot)]$. Indeed distributions are defined as follows:
Definition A.5 A distribution on \((a, b)\) is an equivalent class in the set of all fundamental sequences on \((a, b)\).

For instance, the Dirac distribution has to be seen as the limit of a sequence of continuous functions \(\delta_n(\cdot)\) with support \(K_n = [t_c, t_c + \Delta t_n]\), \(\Delta t_n \to 0\) as \(n \to +\infty\), and:

- i) \(\delta_n(\cdot) \geq 0\)
- ii) for all \(n \in \mathbb{N}\), \(\int_{-\infty}^{+\infty} \delta_n(\tau) d\tau = 1\)
- iii) for all \(a > 0\), \(\lim_{n \to +\infty} \int_{|\tau| > a + t_c} \delta_n(\tau) d\tau = 0\)

Then \(\delta_n(\cdot) \to \delta_{t_c}\) as \(n \to +\infty\), and we know that there exists a sequence of continuous functions \(P_n(t)\) and \(k\) such that \(P_n(t) = \delta_n\), and \(\{P_n\}\) converges uniformly. Note that such a fundamental sequence (called a delta-sequence) that determines the Dirac measure is by far not unique. Examples of functions \(\delta_n\) that belongs to this equivalent class are given in [12] chapters land 2:

\[ 6(t) = \left[ \frac{1}{\sqrt{2\pi n}} \exp \left( -\frac{n z^2}{2} \right) \right] = \left[ \frac{1}{\pi \exp(nz) + \exp(-nz)} \right]. \]

Then \(\delta_n(\cdot)\) are smooth, Delta-sequences are also called mollifiers [71] p.70. They have the property that for any continuous function \(f\), then the convolution \(\delta \ast f \to f\) uniformly on any compact \(\subset \mathbb{R}\).

In the sequential formulation one also retrieves that distributions have derivatives of any order:

Definition A.6 If a fundamental sequence \(\{f_n(\cdot)\}\) consists of functions with continuous \(m\)th derivatives (i.e. \(f_n \in C^m(a, b)\)) then the distribution \([f_n^{(m)}(\cdot)]\) is the \(m\)th derivative of the distribution \([f_n(\cdot)]\).

Each distribution has derivatives of all orders as one can always choose a fundamental sequence of functions which are differentiable up to an arbitrary order. For instance if \(f_n(x) = \sqrt{\frac{n}{2\pi}} \exp \left( -\frac{n x^2}{2} \right)\), \([f_n^{(m)}(\cdot)] = \delta_{t=0}^{(m)}\).

The product of two distributions \(T_1\) and \(T_2\) has a meaning if and only if the sequence \((T_1 \ast \delta_n)(T_2 \ast \delta_n)\) possesses a limit for any delta-sequence \(\delta_n\).

We encountered in chapter 1 expressions involving the product \(\delta^2_0 \triangleq \delta_0 \delta_0\), and we noted that it has no meaning [111] [12] §12.5. In fact, it can sometimes be given a meaning, as shown in [12] §14.3. For instance, the object \(\delta_0^2 - \frac{1}{x^2} \left( \frac{1}{x} \right)^2\) which at first sight has no meaning can be shown to define a distribution equal to \(-\frac{1}{x^2} \frac{1}{x^2}\). Note that \(\frac{1}{x^2}\) is different from \(\left( \frac{1}{x} \right)^2\) when considered as distributions [12] §13.2. Indeed,

\[ \left( \frac{1}{x} \right)^2 = \lim_{n \to +\infty} \left( \delta_n \ast \frac{1}{x} \right)^2 \to +\infty, \]

whereas \(\frac{1}{x^2} = -\frac{d^2 \ln |z|}{dz^2}\). See also [471] concerning the product of Dirac distributions.
A.3 Notions of convergence

Some existential results presented in chapter 2 used the notion of strong and weak* convergence. Let us explain here what this means. For simplicity we restrict ourselves to the spaces $\mathcal{D}$ and $\mathcal{D}^*$. The definitions exist also for other spaces of functions and their dual spaces like Sobolev spaces, see e.g. [71].

**Definition A.7** A sequence of functions $\varphi_n \in \mathcal{D}$ is weakly convergent to $\varphi \in \mathcal{D}$ if for each $T \in \mathcal{D}^*$ one has

$$\lim_{n \to +\infty} \langle T, \varphi_n \rangle = \langle T, \varphi \rangle$$

(A.2)

The sequence $\varphi_n \in \mathcal{D}$ is convergent to $\varphi$ in the topology of $\mathcal{D}$ if their supports are contained in a fixed compact set, $\varphi_n \to \varphi$ uniformly and all derivatives $\varphi_n^{(k)} \to \varphi^{(k)}$ uniformly, for all $k \geq 1$.

**Definition A.8** A sequence of functionals $T_n \in \mathcal{D}^*$ is weakly* convergent to $T \in \mathcal{D}^*$ if for each $\varphi \in \mathcal{D}$ one has

$$\lim_{n \to +\infty} \langle T_n, \varphi \rangle = \langle T, \varphi \rangle$$

(A.3)

It is also possible to define a strong convergence in $\mathcal{D}^*$. However in $\mathcal{D}$ and $\mathcal{D}^*$, strong and weak convergence coincide [478] [182] §6.3, theorem 2.

The notation weak* is to recall that this applies to elements of $\mathcal{D}^*$. As an example, consider the sequence of functions $f_n(x) = n \cos(nx)$. Clearly $\sup_{x \in \mathbb{R}} |f_n(x)| = n$ so that this sequence does not converge uniformly. However $\langle f_n, \varphi \rangle = -\frac{1}{n} < \cos(nx), \varphi > \to 0$ as $n \to +\infty$. Hence $\{f_n\} \to 0$ in a weak* sense.

Such notions are important because in many existential results, one needs compactness of certain spaces. It happens that a space can be compact for a certain notion of convergence, and not for a different one.

**Remark A.3** We have mainly used the space $\mathcal{D}$ and its dual $\mathcal{D}^*$. There are of course many other examples of spaces of functions and of attached dual spaces. For instance if $E = L^p(I)$, $2 \leq p < +\infty$, then $E^* = L^q(I)$, with $\frac{1}{p} + \frac{1}{q} = 1$. In this case $E$ is a normed space (contrarily to $\mathcal{D}$ which is not), and the norm of $\varphi \in E$ is given by the $L^p$-norm. The norm of $\varphi$ can be expressed as $||\varphi|| = \sup_{||T|| \leq 1} | \langle T, \varphi \rangle |$, $T \in E^*$. This second assertion is not a definition, but the result of a theorem [71] p.5. In this example the roles of $p$ and $q$ can be reversed. Weak, strong, and weak* convergence can be defined for such spaces. Sobolev spaces which are described in appendix C are another example of spaces of functions that are often used in functional analysis.
Appendix B

Measures and integrals

We have seen that a proper statement of nonsmooth shock dynamics involves to consider bounded forces as density with respect to the Lebesgue measure $dx$ of the contact impulse measure, whereas contact percussions are atoms of the contact impulse measure, and the impulse magnitude is the density of these atoms with respect to the Dirac measure at the impact time $\delta_{t_k}$. The aim of this appendix is to introduce all these notions.

Let us start by defining abruptly what is meant by a measure [178]:

**Definition B.1** Let $(X, \mathcal{R})$ be a measurable space. A positive measure (or simply measure) on $(X, \mathcal{R})$ is a mapping $\mu : \mathcal{R} \rightarrow [0, +\infty]$ with the following properties:

- $\mu(\emptyset) = 0$.
- $\mu(\cup_{n\geq1} A_n) = \sum_{n\geq1} \mu(A_n)$ for any sequence $\{A_n\}$ of subsets of $\mathcal{R}$, with $A_n \cap A_m = \emptyset$ for $n \neq m$.

Such a mapping that satisfies the second property is called **countably additive**.

What is important to recall at this stage, is that a measure is defined as a function of sets of $X$ that belong to a family of sets $\mathcal{R}$, i.e. it assigns to a set a positive real number. This is logical, since this mathematical notion originated from the practical notion of simply measuring lengths (which cannot be negative, as one imagines) of intervals, or areas of surfaces.

It is clear that in order to make this definition complete, one needs to define what a measurable space is. This is in fact the starting point of measure theory. This is a very abstract mathematical object, just as topological spaces are. The reader should not be afraid if after having read this paragraph, he (or she) still wonders what measurable spaces are. This does not prevent from understanding most of nonsmooth dynamics. It happens however that integration theory relies on such objects. So it seems hard to introduce measures and Lebesgue’s integral without recalling at least basic definitions on measurable spaces and functions. Recall that
a topological space consists of a set $X$, together with a family $\mathcal{O}$ of subsets of $X$ that satisfy certain properties, and are called open subsets. Roughly speaking, a measurable space is a set $X$, together with a family $\mathcal{R}$ of subsets of $X$ that satisfy other (different) properties than open sets. Such a family of subsets is called a $\sigma$-ring [464], or a $\sigma$-algebra [517]. The properties that elements $A, B \in \mathcal{R}$ must satisfy are: the complement of $A$ in $X$ and $A \cup B$ belong to $\mathcal{R}$, and $\mathcal{R}$ is closed under countable unions (i.e. if one takes a sequence of elements of $\mathcal{R}$ (possibly infinite), their union is still in $\mathcal{R}$). In fact, there is unfortunately no possibility to exhaustively describe the elements of such a family $\mathcal{R}$ [178], which would be quite useful for non-mathematical spirits!

Hence $\mathcal{R}$ is a set of subsets of $X$ that satisfy certain properties. Now if $E$ is any set of subsets of $X$, one can show that there is a smallest $\sigma$-ring which contains $E$ (small in the sense of inclusion of sets). Such a smallest element in the set of all $\sigma$-rings defined on $X$ is said to be generated by $E$. There is a particular $\sigma$-ring, called Borel $\sigma$-ring, denoted as $B$. Such a family is constructed by taking all subsets of $\mathcal{R}$ such that if $A, B \in \mathcal{B}$, then $A \cup B \in \mathcal{B}$, their complement $\in \mathcal{B}$, and $\cup_{n \geq 1} A_n \in \mathcal{B}$ whenever $A_n \in \mathcal{B}$. In fact, $\mathcal{B}$ is generated by intervals of the form $[a, b) \subseteq \mathcal{R}$, i.e. $B$ is the smallest $\sigma$-ring over $\mathcal{R}$ that contains the family $E$ of subsets of the form $[a, b)$. Equivalently, a Borel set is obtained by a countable number of operations starting from open sets of $\mathcal{R}$. Each operation consists in unions, intersections and complements. In other words, take a certain number of open intervals of $\mathcal{R}$. Then apply to them a countable (possibly infinite) number of the above operations: you obtain a Borel set, i.e. Borel sets contain elements of the form $(a, b), [a, b], \{a\}, (a, b], (a, b] \cup [c, d) \cap \{e\}, \ldots$. The collection of all Borel sets in $\mathcal{R}$ is the $\sigma$-ring $B$. Then the space $\mathcal{R}$ together with the family $B$ is a measurable space. A set is said measurable with respect to a measure $\mu$ if it belongs to a $\sigma$-ring that belongs to a measurable space (i.e. one has been able to define the measure $\mu$ as in definition B.1). One feature of Borel sets is that they are measurable for any measure $\mu$ appearing in applications.

Remark B.1 In fact it would be preferable to denote $(X, \mathcal{R}, \mu)$ a measurable space, to emphasize that it is attached to a measure $\mu$.

Another example of measurable space is $(\mathcal{N}, \mathcal{R})$ where $\mathcal{R}$ is a $\sigma$-ring of subsets of $\mathcal{N}$, and the measure is defined as $\mu(A) = (\text{the number of elements of } A)$, with $A \subseteq \mathcal{R}$.

Now we are ready to introduce what is called the Lebesgue measure:

Theorem B.1 ([178]) There exists a unique measure $\lambda$ on the measurable space $(\mathcal{R}, B)$ such that $\lambda([a, b]) = b - a$ for all couples $(a, b)$ of real numbers, with $a \leq b$.

What this theorem says is that after having built and developed all the abstract notions related to measures and measurable spaces, one is able to prove that there is only one measure that corresponds to the intuitive notion of the length of an interval. This is quite reassuring.
Before introducing the Lebesgue's integral, let us recall what is meant by a measurable function

**Definition B.2** Let $(X, \mathcal{R})$ be a measurable space, and $Y$ a topological space. Then $f : X \to Y$ is *measurable* if for all open subsets $B \subseteq Y$, the set $f^{-1}(B)$ belongs to $\mathcal{R}$.

Recall that a set which belongs to $\mathcal{R}$ is called measurable. Hence a function is measurable if its inverse sends any open set into a measurable set. It is important to note that measurability is characterized by starting from the image space $Y$. This we shall retrieve in the definition of Lebesgue's sums below. This allows to treat functions which seem very complex when considered from the source space, but rather simple when considered from the image space.

**Example B.1** For instance, let us consider the following wellknown example $f : [0, 1] \to [0, 1], f(x) = 0$ if $x$ is rational, $f(x) = 1$ otherwise. It seems that $f$ is very irregular, since it is everywhere discontinuous, hence not Riemann integrable (or not Riemann measurable) [178] p.4. But $f$ is a very simple measurable function. Indeed consider for instance the set $f^{-1}\left((\frac{1}{2}, \frac{1}{2})\right)$: this is the set of rational numbers on $[0, 1]$. Such a set is negligible (it is countable\(^1\)), so it is measurable. One can take other examples and check measurability in all cases. It is a simple matter to prove that this function is not Riemann integrable [178]: indeed for any subdivision of $[0, 1]$, as fine as desired, the difference between the Riemann’s sums is always 1, due to density of rationals in $\mathbb{R}$.

**Remark B.2** Since this function $f$ is measurable, it defines a distribution. Thus it admits a *generalized* derivative (\(^2\)), or a derivative in the sense of distributions, defined as $\langle f'(x), \varphi(x) \rangle = -\int_{K_\varphi} f(x)\varphi(x)dx$. Apparently it is not easy to visualize what such a $f'$ is: recall that this is not a function. However since $f \equiv 1$ (except on a countable, hence negligible set), one has $\langle f', \varphi \rangle = -\int_{K_\varphi} \varphi(x)dx = 0$ because $\varphi$ is zero outside $K_\varphi$. Thus $f'$ is the zero distribution.

**Remark B.3** Let $f : I \subseteq \mathbb{R}^n \to Y$, $Y$ a metric space. Then $f$ is measurable if there exists a sequence of simple functions that converges to $f$ Lebesgue a.e..

As a matter of fact, and to stress that all these notions have a purely theoretical interest, one can only prove that there *exists* nonmeasurable functions [479]. But it is not possible to explicitly construct them. One may intuitively understand such a fact by considering that, for instance, the Borel sets mentioned above contain

\(^1\)Recall that a set is said to be *countable* when one is able to associate to each one of its elements an integer $n \in \mathbb{N}$. In other words, there is a bijection between the set and $\mathbb{N}$.

\(^2\)Some authors [562] [400] introduce the notion of distributions through so-called generalized derivatives.
Lebesgue’s integration theory relies on measure theory. Let us introduce the relationship between measures, which are notions attached to sets, and integrals which are notions attached to functions. To begin with, let us consider a bounded and positive simple function \( u : X \to [0, +\infty] \), \( u = \sum_{i=1}^{n} c_i \chi_{C_i} \), where \( \{c_1, \cdots, c_n\} \) is the (finite) set of values taken by \( u \), and \( C_i = u^{-1}(c_i) \). The integral of \( u \) is defined as the number

\[
\int u \triangleq \sum_{i=1}^{n} c_i \mu(C_i) \tag{B.1}
\]

where \( \mu \) is a measure. Note that this definition \textit{a priori} attaches the integral to a particular measure. Consider now a positive function \( f : X \to [0, +\infty] \). Then the integral of \( f \) is given by

\[
\int f \triangleq \sup_{u \in \mathcal{U}} \left( \int u \right) \tag{B.2}
\]

where \( \mathcal{U} \) is the set of simple functions \( u : X \to [0, +\infty] \) with \( u \leq f \). One generally denotes the integral of \( f \) as \( \int f \, d\mu \) when \( \mu \) is not the Lebesgue measure. One generally denotes the Lebesgue measure as \( \lambda[\cdot] \), or as \( dx \), and the Lebesgue integral of a measurable function \( f \) on \([a, b]\) as \( \int_{[a,b]} f \, d\lambda \) or as \( \int_{a}^{b} f(x) \, dx \). All these notations denote the same object. Also the Lebesgue measure (or the length) of an interval \([a, b)\) is the number \( \lambda([a, b)) = \int_{\mathcal{R}} \chi_{[a, b]}(y) \, dy = \int_{a}^{b} \chi_{[a, b]}(x) \, dx \), where \( \chi_{[a, b]} \) is the characteristic function of the interval \([a, b)\) (that is measurable if the interval is, since \( \chi_{[a, b]}^{-1}(I) \) is either \( \emptyset, [a, b) \) or \( \mathcal{R} \) depending on \( I \)).

It could seem at first sight that the consideration of positive functions is restrictive. It is not, because every function can be decomposed into its positive and negative parts \( f^+ \) and \( f^- \), with \( f = f^+ - f^- \). Then \( f \) is integrable if and only if \( f^+ \) and \( f^- \) are. In other words, \( f \) is \( \mu \)-integrable if the number \( \int |f| \, d\mu \) is finite.

Concerning the above example B.1, \( \int_{[0,1]} f(x) \, dx = 1 \). This simple example is interesting because it really shows the essence of Lebesgue’s integral. In a sense, \( f \) "splits" the interval \([0, 1]\) into an infinity of parts, so that it is quite impossible to represent its graph (Notice that if the function was discontinuous only at a countable number of points, it would be possible to look at its graph closely enough, so that the jumps appear. Here this is not possible). Lebesgue’s basic idea is to assign a notion of measure (length) to these parts, and to define the integral from this measure.

Remark B.4 Another way to introduce Lebesgue’s integrals is through the so-called \textit{Lebesgue’s sum} [178] p.141, [517] §3.2. Let \( a, b, c, d \in \mathcal{R} \), \( a < b \), \( c < d \), and \( f : [a, b] \to \mathcal{R} \) be Lebesgue measurable, with \( c \leq f(x) \leq d \). The Lebesgue’s sums are defined as follows: Let \( S = (x_0, x_1, \cdots, x_n) \) be a subdivision of the interval \([c, d]\), i.e. \( c = x_0 < x_1 < \cdots < x_n = d \). Note that one subdivides the \textit{image} space,

\footnote{Recall that \( f^{-1}(I) = \{x : f(x) \in I\} \).}
contrarily to Riemann’s sums where the source space is divided. Define Lebesgue’s sums as the numbers

\[ \sigma(f, S) = \sum_{i=0}^{n-1} \lambda \left[ f^{-1}(\{x_i, x_{i+1}\}) \right] x_i \]

\[ \Sigma(f, S) = \sum_{i=0}^{n-1} \lambda \left[ f^{-1}(\{x_i, x_{i+1}\}) \right] x_{i+1} \] (B.3)

Then as the subdivision step tends towards 0, one gets that \( \sigma(f, S) \) and \( \Sigma(f, S) \) both tend to \( \int_{[a,b]} f(x) dx \). \( \lambda = dx \) denotes the Lebesgue measure. Hence one may say that Riemann’s integral is defined from cutting the source space (i.e. \([a, b]\)) into vertical strips, whereas Lebesgue’s integral is rather defined from cutting the image space (i.e. \([c, d]\)) into horizontal strips.

Notice also that for a Riemann integrable function, the Lebesgue and Riemann integrals are equal.

The density of a measure is defined as follows [178] p.145

**Definition B.3** Let \( \mu \) be a measure, and let \( g \) be \( \mu \)-integrable function. Let \( \mu \) be defined as \( \mu(E) = \int_E f(x)dx \) for any measurable set \( E \), and for some Lebesgue measurable function \( f : \mathbb{R} \rightarrow [0, +\infty] = \mathbb{R}^+ \). Then \( f \) is called the density of \( \mu \) and

\[ \int g d\mu = \int g(x) f(x) dx \] (B.4)

If \( f \) is continuous, then the function \( F : x \mapsto \int_0^x f(t) dt \) is differentiable and \( \frac{dF}{dx} = f(x) \). One denotes the measure \( \mu \) as \( dF \) or as \( \mu_f \). Notice that \( F \) is nondecreasing which is necessary for \( dF \) to be a measure.

For instance in chapter 5, section 5.3, we saw that the sweeping process can be formulated with respect to any measure. It was then necessary to define the densities of the Lebesgue and the Stieltjes measures \( dt \) and \( du \) with respect to a measure \( \mu \). These densities are denoted as \( t'_\mu \) and \( u'_\mu \) respectively. In view of the definition B.3, these notations means that for some function \( f \) as in definition B.3, \( d\mu = f dt \) so that \( t'_\mu(t) = \frac{dt}{d\mu} = \frac{1}{f(t)} \), whereas \( u'_\mu = \frac{du}{d\mu} = \frac{\sigma_u(t)}{f(t)} \) for \( t \neq t_k \), and \( u'_\mu(t_k) = \frac{1}{f(t_k)} \sigma_u(t_k) \).

Let us consider some examples of measures:

- If \( f(x) = 1 \), then \( dF = dx (= \lambda) \), \( dF \) is the Lebesgue measure.
- If \( f(x) \) is a continuous function, \( dF \) is a Lebesgue-Stieltjes measure. The length of an interval \([a, b]\) is equal to \( \int_{[a,b]} dx \) (i.e. the Lebesgue integral of its characteristic function). The Lebesgue-Stieltjes integral generalizes the notion of length to \( \int_{[a,b]} f(x) dx = F(b) - F(a) = \mu_f ([a, b]) \geq 0 \) (This is assured because \( f \) takes nonnegative values, see definition B.3).
- The Dirac measure \( \delta_0 \) is the distributional derivative and Stieltjes measure of the Heaviside function \( h(\cdot) \), i.e. \( \delta_0 = dh = h \). In fact if \( \varphi \in \mathcal{D} \), then
< \delta_0, \varphi > = \varphi (0) \text{ (by definition) and:} \\
< h, \varphi > = - \int h(t) \varphi (t) dt = - \int_0^{+\infty} h(t) \varphi (t) dt = \varphi (0) \quad (B.5) \\
Looking at the equation from another angle, we know that \\
\delta_0 ([a, b]) = \begin{cases} 
0 & \text{if } \{0\} \notin [a, b) \\
1 & \text{if } \{0\} \subseteq [a, b) 
\end{cases} \quad (B.6) \\
To find out what is the integral of a continuous function \( f \) with respect to \( \delta_0 \), we approximate it by a sequence \( \{u_n\} \) of step functions such that \( u_n \to f \) pointwise. From [464] theorem 11.30, if \( u_n(x) = c_{n,i} \) on \([c_{n,i}, c_{n,i+1})\), one gets \\
\int u_n d\delta_0 = \sum_{i=0}^{m} c_{n,i} \delta_0 ([c_{n,i}, c_{n,i+1})) = c_{n,k}, \text{ where } 0 \in [c_{n,k}, c_{n,k+1}). \text{ Since } u_n(0) = c_{n,k} \to f(0) \text{ we conclude that:} \\
\int f d\delta_0 = \lim_{n \to +\infty} \int u_n d\delta_0 = \lim_{n \to +\infty} c_{n,k} = f(0) \quad (B.7) \\
as expected. This is therefore one of the ways in which Dirac measures can be introduced, besides the distributional interpretation as the limit of fundamental sequences of continuous functions, see section A.2.

Remark B.5 A measure \( \mu \) possesses an atom at \( x \in X \) if \( \mu(A) > 0 \) for any set \( A \subset B \) which contains \( x \). If outside the atoms \( \mu = 0 \) (i.e. for any set \( A \) containing no atoms, \( \mu(A) = 0 \)), then \( \mu \) is said to be purely atomic. In other words, a measure is purely atomic if it is concentrated on a countable set. For instance, the Dirac measure or any sum of Dirac measures like contact impulses are atomic measures, which possess atoms at the impact times \( t_k \). A measure with no atoms is said non-atomic, i.e. \( \mu(\{x\}) = 0 \) for all point \( x \in X \). An atomic \( \mu_\alpha \) and a non-atomic \( \mu_{na} \) measures are always mutually singular, i.e. there exists a set \( B \in B \) with complement \( C_B \) such that \( \mu_\alpha(B) = \mu_{na}(C_B) = 0 \).

Remark B.6 It is also possible to define real measures, which are not necessarily positive. They are called signed measures. This is the reason why some authors make it clear that they use positive or nonnegative measures (see chapter 5, section 5.3). One may think for instance of a load \( m(x) \) with changing sign: then the measure of a set \( A \) can be given by the Lebesgue integral of \( m(x) \) over \( A \): one then obtains a measure of density \( m \), denoted as \( mdx \). The definition of such measures is similar to that of positive measures, but the subsequent theory is different.

zero-measure, almost-everywhere

There is a notion that is often used and which deserves to be defined: when one says that some property is true almost everywhere (a.e.). A set \( A \) is of zero-measure
(or $\mu$-negligible, or of $\mu$-measure 0) if it is contained in a measurable set $B$ with $\mu$-measure 0 (i.e. $\int \chi_B d\mu = 0$). A property is true $\mu$-a.e. if the set for which the property fails has $\mu$-measure 0. A function is a.e. zero if the set $f^{-1}(0)$ is the complement of a zero-measure set, and two functions $f$ and $g$ are $\mu$-a.e. equal if $\int |f - g| d\mu = 0$. Note that a countable set has Lebesgue measure zero: for instance the set of rational numbers in $\mathbb{R}$. Sets of measure zero may be different for different measures.

In $\mathbb{R}^n$, a set $A$ has zero Lebesgue measure if for any $\varepsilon > 0$, there exists a countable union of balls $B_1, B_2, \cdots$ of respective volume $v_i$ such that $\sum_{i \geq 1} v_i < \varepsilon$ and $A \subseteq \bigcup_{i \geq 1} B_i$. The volume of the balls has here its common meaning since the used measure is the Lebesgue's measure. Hence it is calculated from their characteristic functions $\chi_{B_i}$ as $v_i = \int_{\mathbb{R}^n} \chi_{B_i}(x) dx = \int_{\mathbb{R}^n} \chi_{B_i} d\lambda$. For another measure $\nu_{i,\mu} = \int_{\mathbb{R}^n} \chi_{B_i} d\mu$ and in general $v_i \neq v_{i,\mu}$.

A necessary and sufficient condition for a bounded function on a bounded interval to be Riemann integrable, is that its set of discontinuity points be of zero Lebesgue measure. Note that although the set of rational numbers is countable, the function in example B.1 is not Riemann integrable, because it is discontinuous everywhere. In fact it can be directly shown [178] p.4 that the Riemann's sums of this function do not converge to the same number.

Measures and distributions

Finally let us end with the relationship between measures and distributions. In fact, although these two notions may appear to be quite different at first sight, there is a fundamental result that relates both, known as Riesz' representation theorem [517] [71]. Roughly, it states that to any linear nonnegative bounded functional $T$ on the space of continuous functions, one can associate a unique measure $\mu$ such that $< T, \varphi > = \int \varphi d\mu$. In another version it follows that a nonnegative distribution $T \in D^*$ (i.e. $< T, \varphi > \geq 0$ for all $\varphi \in D$ with $\varphi \geq 0$) is indeed a measure, as above. This implies that a lot of distributions (although not all) can be represented in the form:

$$< T, \varphi > = \int_{K_{\varphi}} \varphi^{(m)}(x) dg(x)$$

(B.8)

where $g$ is a function of bounded variation (see appendix C), and for some $m \in \mathbb{N}$ [182] §4.3.

Remark B.7 Every measure defines a distribution, but the contrary is not true. For instance, the derivatives of the Dirac distribution (which is associated to the Dirac measure), do not define measures (although they can be written in a from like in (B.8)). Indeed as we have seen above, a measure takes positive values. Hence a measure can be used to construct a signed distribution ($T \in D^*$ is said positive if for all $\varphi \in D$, $\varphi \geq 0$, then $< T, \varphi > \geq 0$). Conversely from Riesz' theorem, a positive distribution defines a measure. But it is clear that $\delta_0$ is not a signed distribution. Since the unilateral constraints impose a sign condition on the contact impulsions, it
is therefore sufficient, to describe nonsmooth impact dynamics, to rely on measures [385].

Detailed expositions of measure and integration theory can be found e.g. in [178] [517] and references therein. In particular a clever and clear exposition of Lebesgue’s measure and integral can be found in [517] §2.0.
Appendix C

Functions of bounded variation in time

C.1 Definition and generalities

Roughly speaking, a function of local bounded variation (LBV) is a function that does not vary too much on any bounded interval of its domain of definition. More rigorously, let \( f(x) \) be a function defined on an interval \( I \). Let \( x_0 < x_1 < \cdots < x_n \) be any subdivision \( S_n \) of \( I \). Then \( f \) has bounded variation on \( I \) if

\[
\sum_{i=0}^{n} |f(x_{i+1}) - f(x_i)| \leq C \tag{C.1}
\]

for some bounded constant \( C \). The number \( \text{var}(f, I) = \sup_{S_n} \text{var}(S_n, f, I) \) is called the total variation of \( f(\cdot) \) on \( I \).

Let us provide some intuitive thoughts on such functions, before introducing other equivalent definitions. When \( f \) is simple (i.e. \( f(I) \) consists of a finite set of real numbers, or in other words \( f \) is piecewise constant with finite number of values), then one easily sees that \( f \) is LBV means that the jumps of \( f \) are bounded. On the other hand, if this same \( f \) possesses very large number of discontinuities, one intuitively deduces that the jumps magnitudes necessarily have to become very small. However in the limit, if the number of jumps becomes infinite, most of them have to be almost zero, otherwise a constant \( C \) as in (C.1) cannot be found. Obviously, it is necessary to be more accurate on what is meant by an infinite number of jumps. In fact, a function that satisfies (C.1) can be shown to possess at most a countable set of discontinuity points. This means that one is able to associate an integer \( n \in \mathbb{N} \) to each \( x \in I \) at which \( f \) jumps. In other words, the discontinuity points constitute a sequence \( \{x_n\}, n \in \mathbb{N} \), possibly infinite. It follows that in the case of functions of one variable only, bounded variation implies that the function has only points of continuity or jump points. No other sort of point can be encountered.

The property in (C.1) is equivalent to anyone of the following statements:
i) There exists a constant $C < +\infty$ such that for all $\varphi \in \mathcal{D}$, $| < f, \dot{\varphi} > | = | \int_{K_{\varphi}} f(t) \dot{\varphi}(t) dt | \leq C \sup_{t \in K_{\varphi}} \| \varphi(t) \| (1)$,

ii) $f \in L^1$ and the generalized derivative of $f$ (or equivalently its distributional derivative) is a bounded measure.

iii) There exist two nondecreasing functions $f_1$ and $f_2$ such that $f = f_2 - f_1$.

iv) There exist a continuous function $g(\cdot)$ of bounded variation and a piecewise constant function $s(\cdot)$ (called the jump function) such that $f = g + s$.

Let $I = [a, b]$. A function $f$ of bounded variation in $I$ has right-limits on $[a, b)$, left-limits on $(a, b]$ and is bounded on $I$. Moreover:

v) Every function $\varphi \in C^0(I)$ is Riemann integrable with respect to $f$ and $| \int_I \varphi(x) df(x) | \leq K \sup_{x \in I} \| \varphi(x) \|$ for some $K < +\infty$. Moreover var$(f, I)$ is the smallest $K$ such that the inequality is satisfied.

Remark C.1 The number $\int_I \varphi(x) df(x)$ is called the Riemann-Stieltjes integral of $\varphi$ on $I$ with respect to $df$. Roughly speaking, the Riemann-Stieltjes integral of a function is defined similarly as the Riemann integral. Riemann’s sums are defined as $R(\varphi, I) = \sum_{i=1}^{n} \varphi(\xi_i)(x_i - x_{i-1})$, where the $x_i$’s form a subdivision of $I$ and $\xi_i \in [x_i, x_{i-1})$. Riemann-Stieltjes’ sums are defined as $RS(\varphi, I) = \sum_{i=1}^{n} \varphi(\xi_i)(f(x_i) - f(x_{i-1}))$. Under certain conditions on the functions $\varphi$ and $f$, it can be proved that by taking the supremum of these sums over all possible subdivisions of $I$, one obtains a unique number called the integral of $f$ on $I$.

Remark C.2 From property iv) one deduces that the distributional derivative of $f$ is the sum of three terms: an atomic measure $\mu_a$ which is the derivative of the jump function $s$, a Lebesgue integrable function $\mu$ and a singular nonatomic measure $\mu_{na}$. The sum of $\mu_{na}$ with $\mu df$ is the derivative of $g$.

### C.2 Spaces of functions of bounded variation

From the definition of functions of bounded variation, one can construct a linear space denoted as $BV(I)$ [562] [477]. The space $BV(I)$ consists of scalar functions $f$ on $I = [a, b]$ which are of bounded variation. The norm of $f$ is defined as $\|f\| = \text{var}(f, I) + |f(a^+)$. Equipped with this norm, $BV(I)$ is complete. Similarly, a vector function $f \in \mathbb{R}^n$ belongs to $BV(I)^*$ (it has nothing to do with the $*$ employed to denote dual spaces.) if each one of its components belongs to $BV(I)$. The norm on $BV(I)^*$ is defined as $\|f\|^* = \sum_{i=1}^{n} |f_i|$.

Remark C.3 Notice that not all continuous functions are $LBV$. For instance, the function $f : [0, 1] \to \mathbb{R}$, $f(0) = 0$, $f(x) = x \sin(\pi x)$ is continuous on $[0, 1]$, but it is

1Note that since $\varphi \in \mathcal{D}$, $< f, \dot{\varphi} > = - < f, \varphi >$ so that the inequality can also be written as [517] p.15: $| \int_{K_{\varphi}} \varphi(t) f(t) dt | \leq C \sup_{t \in K_{\varphi}} \| \varphi(t) \|$, if $f$ has a meaning.

2This is the notation employed in [477].
C.3. SOBOLEV SPACES

not of bounded variation. This is easy to see intuitively since as \( x \) approaches zero, \( f \) oscillates infinitely often between \(-1\) and \(+1\).

More details on functions of bounded variation can be found for instance in [383] [562] [517]. The interest of mathematicians for such functions is motivated by the existing physical models which involve solutions of bounded variation. Dynamical systems with state unilateral constraints, shock waves, problems of continuum mechanics \ldots

C.3 Sobolev spaces

Some existential results presented in chapters 2 and 3 use the notion of Sobolev spaces. Sobolev spaces are spaces of functions, defined as follows [71] [118]:

**Definition C.1** Let \( 1 < p \leq +\infty \). The Sobolev space \( W^{1,p}(I) \), where \( I \subset \mathbb{R} \) is an open interval (bounded or not), is the set of functions \( f(\cdot) \) such that

i) \( f \in L^p(I) \)

ii) There exists a function \( g \in L^p(I) \) such that \( \int_I f \varphi = -\int_I g \varphi \) for all \( \varphi \in \mathcal{D} \) whose support is contained in \( I \).

The 1 in \( W^{1,p} \) means that only the first derivatives are taken into account. The \( p \) is for the \( L^p \) norm used in the definition. One recognizes in ii that \( g \) is a generalized derivative of \( f \), or its derivative in the distributional sense (i.e. an element of \( \mathcal{D}^* \)). Hence a function \( f \) belongs to a Sobolev space if its distributional derivative \( g \) can be considered as a \( L^p(I) \) function. In particular this precludes discontinuous functions to belong to \( W^{1,p}(I) \) for any \( 1 \leq p \leq +\infty \), since their distributional derivative cannot be considered as a function (it is a singular distribution).

One can also define Sobolev spaces as spaces of functions such that the norm \( \|f\|_{W^{1,p}} = \|f\|_{L^p} + \|g\|_{L^p} \) makes sense (i.e. in particular \( g \) is a function) and is finite.

When \( 2 \leq p \leq +\infty \), \( f(\cdot) \in W^{1,p}(I) \) if and only if for all \( \varphi \in \mathcal{D} \) with support contained in \( I \), one has \( \|f \varphi\|_{L^p} \leq K \|\varphi\|_{L^2} \), with \( \frac{1}{p} + \frac{1}{2} = 1 \). If \( p = 1 \), then \( f \in W^{1,1} \) implies this inequality, but not the inverse. Also if \( p = 1 \) and if \( I \) is a bounded interval, one sees that the above inequality characterizes functions of bounded variations (see i) in section C.1).

In summary, \( q \in W^{1,1}(I) \) if an only if \( q \) and \( \dot{q} \in L^1(I) \). But this does not mean that \( \dot{q} \in BV(I) \) and that \( \dot{q} \) is a bounded measure. Therefore existential results in Sobolev spaces are quite different from those in the space \( Q \) defined in (3.16), chapter 3. This feature places the dynamics of systems with unilateral constraints clearly apart of many other evolution problems, where Sobolev spaces are used to prove existence of solutions.
Appendix D

Elements of convex analysis

As we have seen throughout the book, an important class of studies devoted to mechanical systems with unilateral constraints use mathematical tools from convex analysis. This is the case for the general sweeping process formulation and the works in [476] [416]. We recall here some basic definitions used in this setting. Roughly speaking, convex analysis is that part of functional analysis, dealing with convex sets and functions. As we noticed in section 5.3, all those mathematical tools aim at generalizing the simple well-known notions of tangent space, normal direction, in order to get a powerful framework to study evolution problems, one of which is the dynamics of systems subject to unilateral constraints. But there are other applications, see the book [382]. Convex analysis is also used in the framework of nonsmooth Lyapunov functions, with Clarke’s generalized derivative.

**Definition D.1** Let $\Phi$ be a closed convex domain $\subset \mathbb{R}^n$, with nonempty interior and smooth boundary. Then the indicator function of $\Phi$ is given by

$$
\psi_\Phi(q) = \begin{cases} 
0 & \text{if } q \in \Phi \\
+\infty & \text{otherwise} 
\end{cases}
$$

The indicator function $\psi_\Phi(q)$ is convex if and only if $\Phi$ is a convex set [374] p.11.

**Definition D.2** The tangent cone to $\Phi$ at $q$ is defined as

$$
T_\Phi(q) = \bigcup_{y \in \Phi} \bigcup_{\lambda > 0} \lambda(y - q)
$$

When $q \notin \Phi$, one normally defines $T_\Phi(q) = \emptyset$

One can check that this definition and definition 5.1 are similar for $q \in \Phi$. The overbar indicates the complementary space. Notice simply that the definitions in 5.1 (used in the sweeping process formulation) and in D.2 (used in [416]) differ by the fact that $-V(q) = T_\Phi(q)$ (hence the corresponding normal cones point either outwards $-N(q)$—or inwards $-N_\Phi(q)$—the domain $\Phi$ for $q \in \partial \Phi$).
Definition D.3 The normal cone to \( \Phi \) at \( q \) is defined as
\[
N_\Phi(q) = \{ y : \forall z \in T_\Phi(q), y^Tz \leq 0 \}
\] (D.3)

The following definition generalizes the notion of a derivative, for indicator functions.

Definition D.4 The subdifferential of \( \psi_\Phi \), denoted as \( \partial \psi_\Phi \), is defined as
\[
\partial \psi_\Phi(q) = \begin{cases} 
0 & \text{if } q \in \text{Int } (\Phi) \\
N_\Phi(q) & \text{if } q \in \partial \Phi 
\end{cases}
\] (D.4)

One sees that on the boundary of the domain \( \Phi \), the subdifferential is equal to the normal cone. In case of a smooth codimension one constraint, this simply reduces to the classical normal direction to the considered surface \( \partial \Phi \) at the considered point \( u \). One may check this graphically, see also figures D.1.

For a locally Lipschitz continuous and convex function \( g : \mathbb{R}^n \rightarrow \mathbb{R} \), the subdifferential of \( g \), i.e. \( \partial g(x) \), is equal to Clarke's generalized gradient of \( g \) at \( x \), given by \[109\]
\[
\partial g(x) = \text{co } \left\{ y : y = \lim_{i \to \infty} \nabla_x g(x_i), \lim_{i \to \infty} x_i = x \right\}
\] (D.5)
The generalized gradient is therefore defined by considering all the possible sequences \( \{x_i\} \) that converge towards \( x \), and such that \( \nabla_x g(x_i) \) exists for all \( i \) as well as the limit \( y \). Then one takes the convex hull of all these limits to construct the generalized gradient. When \( g \) is differentiable with continuous gradient, then the generalized derivative (hence the subdifferential) coincides with the usual derivative (or gradient).

Example D.1 Consider \( f(x) = |x| = x \text{sgn}(x) \), \( x \in \mathbb{R} \). According to the notations employed in chapter 1, we get \( \frac{df}{dx} = \sigma(x(x_k))\delta_{x_k} + \{ \frac{df}{dx} \} \), where \( x_k \) is a point where \( f(x) \) has a jump\(^1\). Here clearly \( f(x) \) is continuous, hence \( \frac{df}{dx} = \frac{d(x \text{sgn}(x))}{dx} = 1 \) or \(-1\) depending on whether \( x > 0 \) or \( x < 0 \) respectively. What happens at \( x = 0 \)? Clarke's generalized gradient of \( f(x) \) is given by
\[
\partial f(x) = \begin{cases} 
\{-1\} & \text{if } x < 0 \\
\{1\} & \text{if } x > 0 \\
[-1, 1] & \text{if } x = 0 
\end{cases}
\] (D.6)

\(^1\)One might be troubled when calculating the derivative of \( f \) since a discontinuous function appears in its definition. Recall however that the generalized chain rule applies as \( f = \{f\} + \sigma_f(x_k)\delta_{x_k} \), from which one concludes since \( \sigma_f \) is zero (\( f(x) \) is a continuous function) that \( \frac{df}{dx} \) is equal to its value outside \( x = 0 \), which is trivially 1 or \(-1\) depending on the sign of \( x \).
where the brackets \{ \cdot \} emphasize that the generalized derivative is a set (may be reduced to points).

To clarify those definitions, we illustrate the tangent, normal cones and the subdifferential on simple cases, see figure D.1. Notice that in the sweeping process formulation presented in chapter 5, the tangent cone $T(q)$ to the admissible space $\Phi$ at $q$ plays the role of the subspace $\Phi$ in definitions D.1 and D.4. Then the subdifferential is a function of the velocity $u$ considered as an element of the tangent cone $V(q)$.

In relationship with some equivalences needed in the sweeping process formulation, let us introduce the following definitions and lemma:

**Definition D.5 (Proximal points [375])** Given a linear space $E \ni x$ and a convex, lower semicontinuous, nonidentically infinite function $f(x)$, the **proximal point** of $z$ with respect to $f$, denoted as $\text{prox}_f z$, is the point where the function

$$u \mapsto \frac{1}{2} ||z - u||^2 + f(u) \quad (D.7)$$

attains its minimum.

If $f(\cdot)$ is the indicatrix function of a closed convex set $C$, i.e. $f = \psi_C$, then $\text{prox}_f z$ is the nearest point from $z$ that belongs to $C$, denoted as $\text{proj}_C z$. Also one has $-x \in \partial \psi_C(u) \iff u = \text{proj}_C(u - \rho x) \iff u = \text{prox}_{\psi_C}(u - \rho x)$ for all $\rho > 0$.

The following definition may be found in [375] [374] [71].

Figure D.1: Tangent and normal cones.
Definition D.6 (Dual function) Let $E$ be a linear space equipped with a metric $\langle \cdot, \cdot \rangle$. To any convex, lower semicontinuous function $f(x)$ not identically infinite, one associates its conjugate (or dual or polar) function $g(y) = \sup_{x \in E} \langle x, y \rangle - f(x)$. $g(\cdot)$ is thus the smallest function for which $f(x) + g(y) \geq \langle x, y \rangle$.

The inequality $f(x) + g(y) \geq \langle x, y \rangle$ when $f(\cdot)$ and $g(\cdot)$ are dual functions, is called the Young's inequality [16] p.64. Notice that it is sufficient that $f(\cdot)$ in definition D.6 be nonidentically $+\infty$ to define its conjugate [71] p.9. But if $f(\cdot)$ is convex and lower semicontinuous, then Fenchel-Moreau theorem (see below) applies. Also the conjugate function of a nonidentically infinite function, is convex and lower semicontinuous.

As an illustration of such dual functions, let us consider the following examples:

- If $f(x) = x^2$, one finds that $g(y) = \frac{y^2}{4}$
- If $f(x) = ||x||$, then $g(y) = \begin{cases} 0 & \text{if } ||y|| \leq 1 \\ +\infty & \text{if } ||y|| > 1 \end{cases}$. Note that this function $g(\cdot)$ is the indicator function of the ball $||y|| \leq 1$.
- If $f(x) = \frac{x^2}{\alpha}$, then $g(y) = \frac{y^2}{\beta}$, with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $\alpha > 1$, $\beta > 1$.

Remark D.1 The dual function of a function $f(\cdot)$ is often denoted as $f^*(\cdot)$.

- Consider the function $f(x) = \begin{cases} -a & \text{if } x < 0 \\ b & \text{if } x > 0 \\ [-a, b] & \text{if } x = 0 \end{cases}$. Consider the indicator function of the interval $[-a, b]$, $\psi_{[-a, b]}(y)$. Its conjugate function is given by $\psi_{[-a, b]}^*(x) = \sup_{y \in [-a, b]} \langle x, y \rangle$ ([374] example 6.c). Then $f(x) = \partial \psi_{[-a, b]}^*(x)$, the subdifferential (or Clarke's gradient) of $\psi(x)$. One recognizes the expression of Coulomb's friction law in the definition of the function $f(x)$. If $x$ is the velocity then the tangential force $F_t$ satisfies $F_t \in \partial \psi_{[-a, b]}^*(x)$ which is equivalent to $-x \in \partial \psi_{[-a, b]}(F_t)$. This is used in the sweeping process formulation to express the dynamical equations in suitable different forms. It is a general result [374] that if $f(\cdot)$ and $g(\cdot)$ are 2 dual functions, then for any $x$ and $y$ in their respective sets, one has

$$
\psi_{[-a, b]}^*(x) = \sup_{y \in [-a, b]} \langle x, y \rangle
$$

These equivalencies may be used to express the same law of motion in various manners, either using a function or its conjugate. Also one deduces that if a function
\( f(x) \) possesses a minimum at \( x_0 \), then one has:

\[
0 \in \partial f(x_0) \iff x_0 \in \partial g(0) \iff f(x_0) + g(0) = 0 \quad (D.9)
\]

which means that \( g \) must be differentiable at \( y = 0 \) for \( x_0 \) to be a minimum point of \( f \), and vice-versa \([374]\) p.60.

- In mechanics, the dual function is obtained through the so-called Legendre transformation \([16]\) §14, which allows one to construct the Hamiltonian function from the Lagrangian, where the Lagrangian is seen as a function of \( \dot{q} \), whereas the Hamiltonian is seen as a function of generalized momentum \( p \). If \( L(q, \dot{q}) = \frac{1}{2}q^T M(q) \dot{q} - U(q) \), then \( H(p, q) = p^T M^{-1} p + U(q) \). It is known that the Legendre transformation is involutive (i.e. when applied twice, it is the identity) \([16]\) §14,C. In fact, the following is true

**Theorem D.1 (Fenchel-Moreau \([71]\))** Assume that \( f(\cdot) \) is convex, lower semi-continuous, and \( f \neq +\infty \). Then \( f^{**} = f \), i.e. the dual function of the dual function of \( f \), is \( f \) itself.

Hence from the example above, one deduces that \( ||x|| = \sup_{y \in E^*, ||y|| \leq 1} | < y, x > | \), which is in fact the definition of the norm of \( x \in E \) \([71]\). In classical mechanics, one obtains that the Legendre transformation of the Hamiltonian is the Lagrangian. It is clear in this context why the function \( g(y) \) (or \( f^*(y) \)) is called the dual function of \( f(x) \), since the Hamiltonian formulation of dynamics involves generalized momenta \( p \), which belong to the dual space \( T^*V_q \) of the tangent space \( TV_q \) of the system's configuration space \( V \) at the point \( q \). \( T^*V_q \) is also called the cotangent space to \( V \) at \( q \) \([16]\) p.202.

Given a cone \( V(q) \), it polar cone \( N(q) \) is defined as \( N(q) = \{ u : \forall v \in V(q), < v, u > \leq 0 \} \), where it is understood that \( u \) belongs to the space dual of that of \( v \) \([379]\). The normal cone in definition D.3 is the polar cone of the tangent cone in definition D.2. The following lemma is useful to write down the different formulations of the sweeping process:

**Lemma D.1 (Lemma of the two cones \([372]\))** If \( V \) and \( N \) denote a pair of mutually polar closed convex cones of a Euclidean linear space \( E \), and if \( x, y, z \) are three points of \( E \), the following assertions are equivalent

- \( x = \text{prox} (V, z), y = \text{prox} (N, z) \)
- \( z = x + y, x \in V, y \in N, x.y = 0 \)

It also follows as a corollary that

\[
x = \text{prox} (V, z) \iff z - x = \text{prox} (N, z) \quad (D.10)
\]
The dot is the scalar product in $E$ ($E = \mathbb{R}^n$ in general for us).

Finally it may be worth recalling the definition of lower semicontinuity of a function that is currently used in functional analysis, especially in variational calculus \cite{118} \cite{71} and convex analysis:

**Definition D.7** A function $f : E \to (-\infty, +\infty]$ is *lower semicontinuous* (lsc) if for all $\alpha \in \mathbb{R}$, the set $S_\alpha = \{x : f(x) \leq \alpha\}$ is closed. In other words, $f$ is lsc at $x_0$ if for all $b \in \mathbb{R}$ such that $b < f(x_0)$, there exists a neighborhood $V$ of $x_0$ such that for all $x \in V$, $b < f(x)$. $f$ is lsc on $E$ if it is lsc for all $x_0 \in E$.

For instance the characteristic function $\chi_I$ of an open interval $I = (a, b)$ is lsc. Indeed since $I$ is open, for any $x \in I$ there exists a ball $B(x, r)$ centered at $x$, of radius $r > 0$, such that for all $y \in B(x, r)$, then $y \in I$ (i.e. $B(x, r) \subset I$). Hence take $b < 1$: clearly for any $x \in I$, it suffices to take $V = B(x, r)$ as a neighborhood of $x$. And if $x \notin I$, and $b < 0 = \chi_I$, it suffices to take $V = \mathbb{R}$. Also the indicator function of a closed convex nonempty domain is lsc \cite{71}. An example is depicted in figure D.2.
Appendix E

Dissipative systems

The material in this appendix is taken from [197] [198] and [86].

Consider a discrete-time dynamical system of the form

\[
x(k + 1) = Ax(k) + Bu(k)
\]

\[
y(k) = Cx(k) + Du(k)
\]

where \(x \in \mathbb{R}^n\), \(u \in \mathbb{R}^m\), \(y \in \mathbb{R}^m\).

**Definition E.1** ([86]) The dynamical system in (E.1) with supply rate \(W(u, y)\) is said to be **dissipative** if there exists a nonnegative function \(V : \mathbb{R}^n \to \mathbb{R}\), \(V(0) = 0\), called the **storage function**, such that for all \(u \in \mathbb{R}^m\) and all \(k \in \mathbb{N}\)

\[
V(x(k + 1)) - V(x(k)) \leq W(u(k), y(k))
\]

(E.2)

which is equivalent to

\[
V(x(k + 1)) - V(0) \leq \sum_{i=0}^{k} W(u(i), y(i))
\]

(E.3)

for all \(k\), \(u(k)\) and \(x(0)\).

If \(W(u, y) = u^T y\), the system is said to be **passive** [198]. If the following inequality is verified

\[
V(x(k + 1)) - V(x(k)) \leq y(k)^T u(k) - S(x(k))
\]

(E.4)

where \(S : \mathbb{R}^n \to \mathbb{R}\) is a positive definite function, the system is said to be **strictly passive**.

The following is the equivalent of the Kalman-Yakubovitch-Popov lemma (see [197] [198]) for discrete-time systems

**Claim E.1** The system in (E.1) is passive lossless with storage function \(V = \frac{1}{2} x^T P x\) if and only if there exists a positive definite matrix \(P\) such that
• $A^T PA = P$
• $B^T PA = C$
• $D + D^T = B^T PB$

Storage functions can also be defined in a variational form [176]

$$V(x(0)) = -\inf_u \sum_{k=0}^{+\infty} W(u(k), y(k))$$  \hspace{1cm} \text{(E.5)}

where the controls $u \in l_{2e}$ drive the system from the initial state $x(0)$. 
Bibliography


[52] L. Boltzmann, 1897 "Vorlesungen ubder die principe der mechanik", T.1 (repr.: J.A. Barth, Leipzig, 1922)


[117] A.A. ten Dam, 1993 "Representations of dynamical systems described by behavioral inequalities", European Control Conference, pp.1780-1783, June, Groningen, NL.


BIBLIOGRAPHY


BIBLIOGRAPHY


BIBLIOGRAPHY


BIBLIOGRAPHY


[528] D. Tabor, 1951 The hardness of metals, Oxford University Press.


[556] T.L. Vincent, 1995 "Controllable targets on or near a chaotic attractor", University of Arizona, Aerospace and Mechanical Engineering.


J. Walker, 1988 "Drop two stacked balls from waist height; the top ball may bounce up to the ceiling", Scientific American, October, pp.140-143.


Y. Wang, 1989 "Dynamics and planning of collisions in robotic manipulation", IEEE Int. Conf. on Robotics and Automation, Scottsdale, AZ, pp.478-483.


E.T. Whittaker, 1904 A treatise on the analytical dynamics of particles and rigid bodies, Cambridge, UK, Cambridge Univ. Press.


[594] B. Yao, M. Tomizuka, 1993 "Robust adaptive motion and force control of robot manipulators in unknown stiffness environment", IEEE Conf. on Decision and Control, pp.142-147, San Antonio, TX, USA.


BIBLIOGRAPHY


[608] L.C. Young, 1980 Lectures on the calculus of variations and optimal control theory, Chelsea publishing company, NY.


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